S101 Units 2 and 3

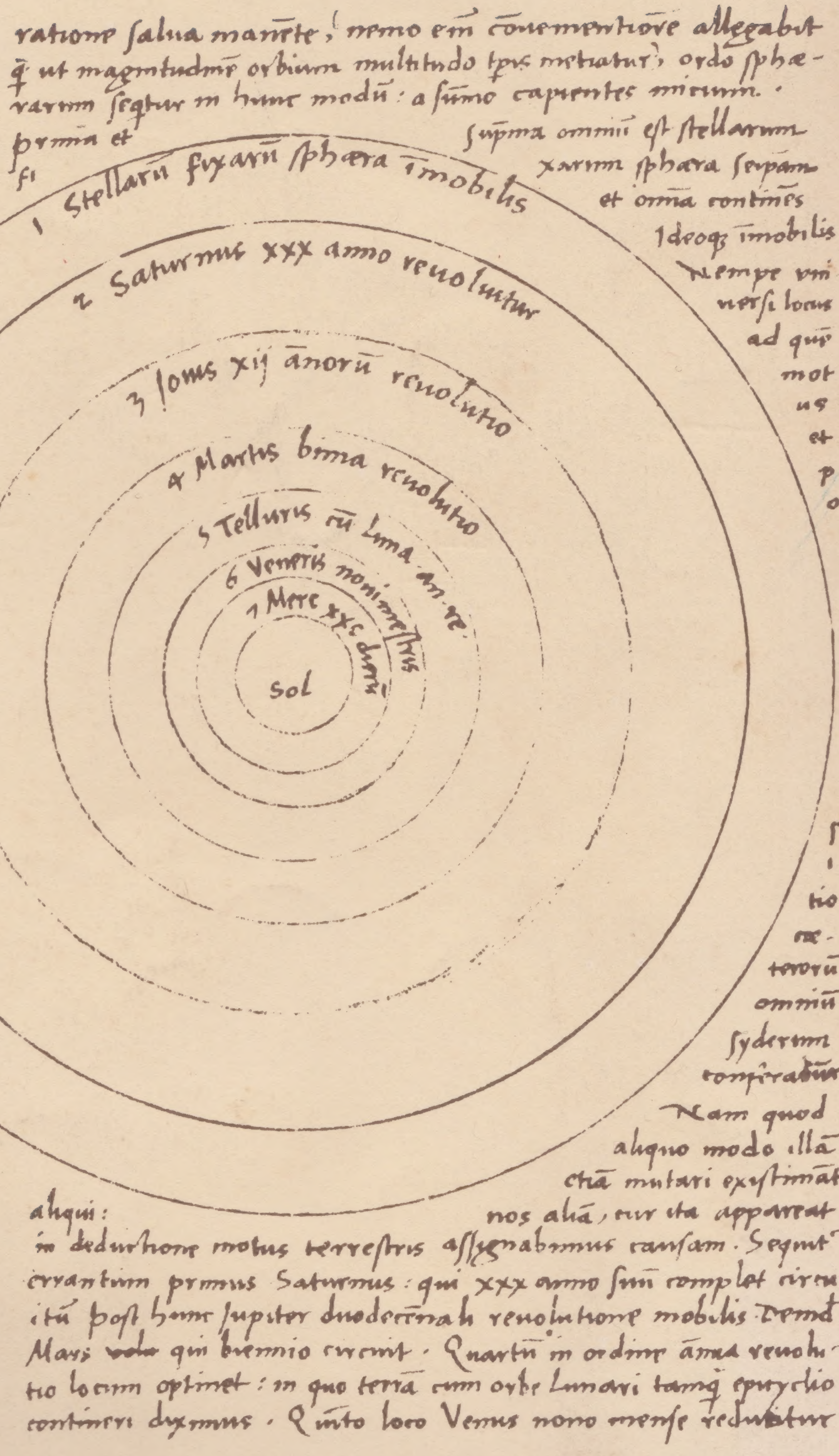
THE OPEN UNIVERSITY

Science : A Foundation Course

2 Measuring the Solar System

3 Motion Under Gravity

A Scientific Theory



The Open University
Science: A Foundation Course

Unit 2
Measuring the Solar System

Prepared by the Science Foundation Course Team

The Open University Press

SCIENCE

S101 Course Team List

A note about the authorship of this text

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2.1

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* See the Comment below Table A in Unit 1.

Study Comment

The Main Text of this Unit comprises the greater part of your week's work. It includes a Home Experiment (Section 2.4).

The TV component includes an important correction to one of the measurements you are asked to make in this Text.

Radio 01, which is also associated with this Unit, explains the concepts of latitude and longitude, which are included in Table A.

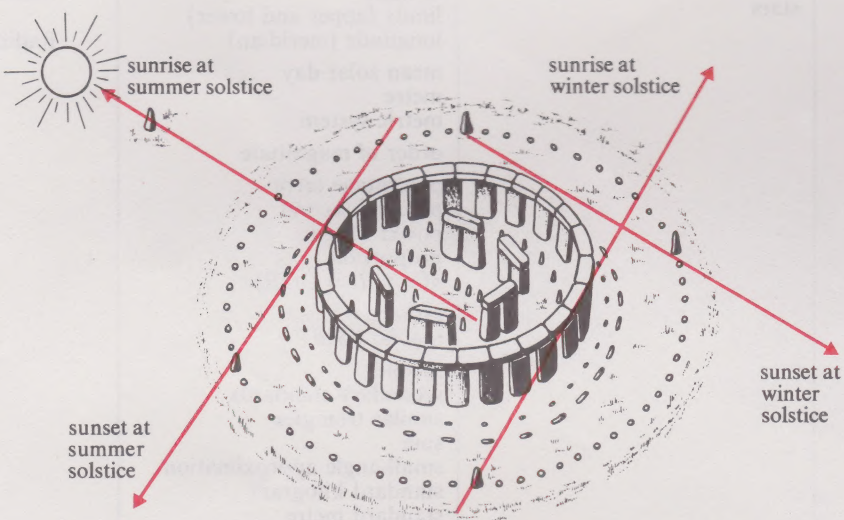


FIGURE 1 Stonehenge. The alignment of various stones gave the position of the rising and setting Sun on the days of summer and winter solstice. Sunrise on the summer solstice must have been of particular importance; the alignment is such that it has to be viewed from the altar stone, inside the monument.

Introduction

This Unit is about measurement—about the assumptions made, the reasoning used, and the limitations involved, whenever you make a measurement. But, in order to understand this topic you must do more than simply listen to what other people have to tell you. You must learn to make measurements for yourself, and you must work to understand why other scientists have made the measurements they have made. That is why, during the course of this Unit, you will be asked to carry out your own Home Experiment, compile tables of results, and criticize constructively other people's measurements. But rather than ask you to learn these techniques and skills for their own sake, we have tried to weave the techniques into the (we think) fascinating story of the early discovery of the 'rules' governing the solar system. In this way, we hope you will see the need to master these skills, and be encouraged and motivated to work at them.

Be warned, however, it won't be easy. Making and interpreting measurements often involves using subtle mathematical reasoning. For instance, a *direct* measurement of the distance between the Earth and the Sun is not possible. So an *indirect* approach must be adopted. This may well mean measuring angles (so you must know about angular measure) and then deducing the distance using a 'triangulation' method (so you must know something about the properties of triangles). These mathematical tools are introduced as and when they are needed. You may feel that the explanation of them in this text is quite adequate for you. If so, you probably won't find this Unit too difficult. But if you're not very familiar with this mathematics, please make sure that you take the advice, given at various places throughout the text, to refer to the explanations and examples in the relevant Sections of *MAFS**.

You may be wondering why it is necessary to bother with this *quantitative* aspect of science at all. Why not just concentrate on the ideas and concepts of science? Well, in some areas we shall do just that. Most of Unit 1 was *qualitative* in approach, and there was plenty of 'good' science in that. But most science can't stop at the qualitative level. Sooner or later, two or three qualitative explanations of a phenomenon will come into conflict. And if science is to be a little more objective than the 'my-explanation-is-as-good-as-yours' argument will allow, there has to be some way of choosing between these rival explanations. That is where measurement becomes important.

Of course, it is not necessary that the sole motivation for making measurements should be the desire to choose between one scientific model and another! Indeed, people have long realized that there are more down-to-earth reasons why they should be able to measure things. How else, for instance, could they know the quantity of grain they were selling, or the distance to market, or the length of time it would take to walk there? No, the need to measure things existed long before the need to differentiate between contending theories of the solar system. One consequence of this is that we have inherited a large array of measurement standards. Naturally, these standards have evolved during the history of mankind, from the rather crude and variable measures of primitive man, to the highly precise measurement standards used in modern science. We no longer measure distances in terms of a man's stride; we now have length standards that are accurate to 1 inch in 10 000 miles. And the immensely impressive 'stone calendar' at Stonehenge (Figure 1) has been superseded by an atomic clock which operates with an accuracy of 1 second in 30 000 years. Indeed, this evolution of measurement standards has been absolutely essential for the development of science. For without a system of well-defined units and precise, reproducible standards, scientists would find it impossible to communicate their findings to each other unambiguously.

So you can see that—if you are to learn about the measurements that have been important in the development of science—it is essential that you become acquainted with the international system of units and standards in present-day use. That is why the whole of Section 1 of this text is devoted to a discussion of these standards. However, you should *not* try to memorize all the details given

* The Open University (1978) S101 *Mathematics for the Foundation Course in Science* (*MAFS*), The Open University Press. Please refer to the S101 *Introduction and Guide* for advice on how to use *MAFS*.

in this Section; indeed many of the concepts will occur again, in more detail, in later Units. Instead, you should view this Section as providing background information to help you put your understanding of subsequent measurement techniques in the context of modern science and technology.

The story of the 'measurement of the solar system' (which takes up the rest of the text) is divided into three parts. The first part (Section 2) describes how the early Greek astronomers estimated the sizes and distances involved in our Sun–Earth–Moon system. Section 3, on the other hand, is basically concerned with measurements involving the *planets* of the solar system, and in particular with the pioneering work of Copernicus, Tycho Brahe, and Kepler. The culmination of this work was the formulation, by Kepler, of his 'three laws of planetary motion'—three laws that succinctly summarized the *regularity* of the motion of the planets of the solar system. An epilogue to this story of Kepler's discovery of his three laws is provided in Section 4. Galileo, using the newly invented telescope, discovered four moons circulating around Jupiter. Kepler's laws were applicable to the planets orbiting the Sun. Could they also be applied to the moons orbiting Jupiter? If so, what was the 'mechanism' behind these laws?

You will not find the answer to this last question in Unit 2. The 'explanation' for Kepler's law was provided by Newton. We shall discuss Newton's contribution in Unit 3.

1 Measurement: units and standards

1.1 What is measurement?

Measurement can usually be reduced to a comparison of something of interest with some agreed standard. So, for instance, if you were to take the width of this book as a unit of length, the lengths of all other objects could be related to the width of the book. One object may be twice as long as the width of the book (i.e. two 'book-widths' long); a second may be three-and-a-half times as long (3.5 'book-widths'); a third may be a quarter as long (0.25 'book-widths'). Once the unit is established, measurement becomes simply a matter of counting. So, if you were told that the Walton Hall boiler-house stands 60 'book-widths' high, you would have some idea of how tall it is, even though you may never have seen it! But note that it was important for you to know what the unit of measure was; furthermore, you even had access to a 'copy' of it. Someone who did not know what the unit of measure was, or who did not have access to the 'standard' book-width, would have no idea what was meant by a height of 60 units. We need standards, and these standards must be known by everyone with whom we wish to communicate. But what standards should we choose?

1.2 Length

Early standards of length tended to be related to the size of the human body. A 'cubit' was the distance between the elbow and the end of the middle finger. The width of a man's hand was used to express the height of a horse in 'hands'. The width of the thumb became an 'inch', the length of the foot became a 'foot'. When measuring cloth, it was convenient to define the distance from the nose to the end of the middle finger when the arm was outstretched as a 'yard'.

These 'personal' standards, however, though conveniently portable, can vary considerably from person to person. Get two drapers together, for instance—a tall one and a short one—and, with this definition of the yard, one of them sells you 'cheaper' cloth! Clearly, we need a greater degree of standardization than this. So what do we do? We invent the *yardstick*. Copies of this stick can now be sent all over the country, with the result that a yard of cloth in Edinburgh is more or less the same length as a yard of cloth in London. But what do we mean by 'more or less?' To within 1/10 inch? Nobody is likely to grumble about that sort of uncertainty if they are buying cloth; but in science we may well want to measure to smaller subdivisions than that. So the 'preciseness' of our standard must improve. The ultimate limit to the accuracy with which we can make a measurement (no matter how good the equipment is) will always be determined by the uncertainty in the standard. It is no good giving a measurement to an accuracy of 1/100 inch, if the agreement as to what constitutes a standard yard

is only good to 1/10 inch. (Mind you, once you have found a way of measuring to an accuracy greater than that to which the standard is defined, the onus is on you to petition for a new standard based on your improved technique!) Hence, as scientists have sought to make more and more accurate measurements, so they have also had to devise more and more precise measurement standards.

Nowadays, there is also a further requirement: our measurement standards should be international. Clearly, this was unimportant when different communities did not interact with each other. Today, however, if scientists in Europe are to understand the measurements of scientists in America, or Russia, or Japan, they must all be acquainted with each other's standards of measurement. Better still, they should all use the same standards. The system of length measurement which the scientific community has agreed to adopt (and with which you will be primarily concerned in this Course) is the system based upon the metre—the *metric system*.

metric system

The history of this system dates back to Napoleonic France at the beginning of the nineteenth century. The *metre* was then originally defined as one ten-millionth of the distance from the Equator to the North Pole along a meridian passing through Dunkirk and Barcelona, and hence also passing very close to the centre of Paris (Figure 2).

metre

FIGURE 2 The metre was originally defined as one ten-millionth of the distance from the pole to the Equator. A team of French surveyors was engaged to measure the distance between Dunkirk and Barcelona. The length of the full quadrant was then determined from astronomical measurements of latitude.

This choice of standard was a very 'safe' one—the standard could not be lost! But it could hardly be called practical. So in 1889 it was officially decided to define the metre as the distance between two parallel marks inscribed on a particular bar made of the metal alloy platinum–iridium. (This alloy was chosen because of its exceptional hardness, and its resistance to corrosion.) Furthermore, to ensure the reproducibility of measurements of this length, the bar had to be kept under specific conditions. It had to be supported in a particular way, for instance, so as to avoid unsuspected deformations, and it had to be kept at the temperature of melting ice so as to prevent the expansion or contraction that would occur if the temperature were allowed to vary. This *standard metre* still exists. It is housed in the International Bureau of Weights and Measures in Sèvres, near Paris. Copies of this bar (i.e. secondary standards) have been made and distributed to national standards offices throughout the world.

standard metre

secondary standards

Even this, however, was not completely satisfactory. For although the length of an object could be compared with the standard metre to a precision of about two parts in ten million (by using a high-powered microscope to view the finely inscribed marks on the metre bar), this precision was still inadequate

for some scientific purposes. (Don't forget, the precision to which we know the standard must be better than the *most* accurate measurement we wish to make.) In addition, making comparisions with a bar that had to be kept under specific conditions in a standards laboratory was still inconvenient. What was required was a standard that anyone (or at least, any scientist) could have access to in his own laboratory, a standard that did not require copies to be made (and hence eliminated the problem of exactness of the copy), and a standard which could be relied upon never to change.

Such a standard now exists. In 1961, by international agreement, a *natural unit of length* was defined, based on the wavelength of a particular orange-red radiation emitted from the krypton gas in a krypton lamp (Figure 3)*. We call this a *natural* unit since it depends on a particular property of nature—namely, the fact that all atoms of a particular species are identical, and consequently emit identical radiations. In fact, the krypton lamp is a bit like the sodium street lamp, in the sense that no matter where the street lamp is situated—London or Glasgow, New York or Helsinki—it still emits the same yellow-coloured light. The colour of the light is determined by the 'structure' of the sodium atoms—and sodium atoms are the same the world over. Naturally, the same is also true of krypton atoms.

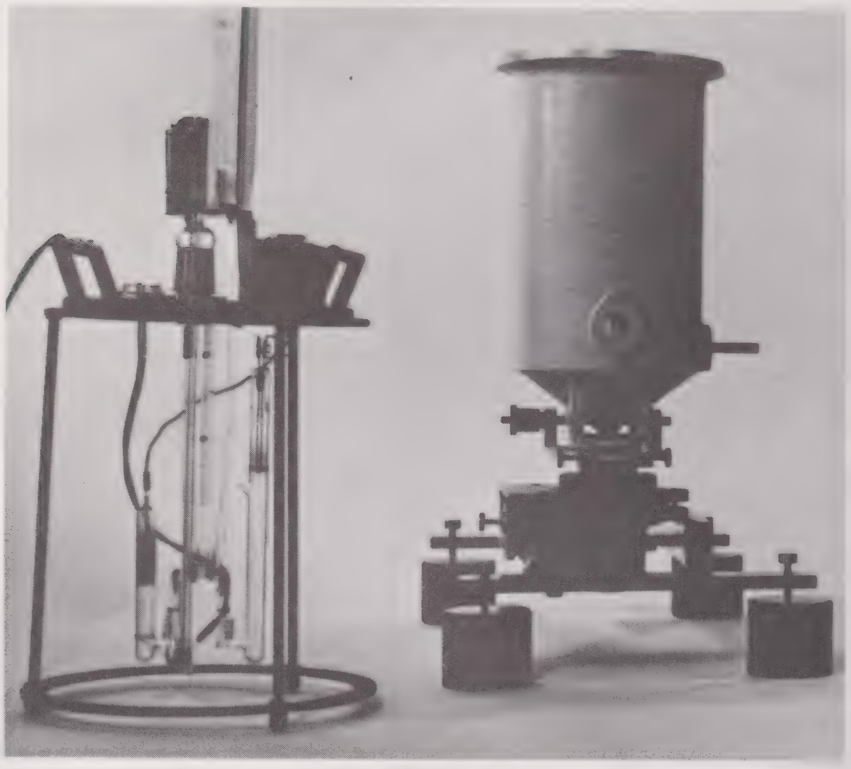


FIGURE 3 Orange-red light from a krypton lamp, such as this (shown here dismantled from its housing), is used to define the international standard of length.

So the length of the standard metre bar was carefully measured in terms of this wavelength of 'krypton light' and it was agreed that exactly 1 650 763.73 wavelengths constituted one metre. This number was chosen so that the old definition of the metre (as the distance between the two scribe marks on the platinum–iridium bar), and the new definition of the metre (as a particular number of wavelengths of krypton light) were kept in agreement. The advantage of the new definition is that it provides a standard of length far more precise than the metre bar. In addition, the krypton standard is readily available to laboratories all over the world, since krypton is present, albeit in small amounts, in the Earth's atmosphere.

Incidentally, you may be interested to know that the inch, the foot and the yard are now *defined exactly* in terms of the metre**. The conversion factors are given in Table 1. In this sense, the krypton wavelength is also the standard of length in the British system as well!

TABLE 1

1 inch = 0.025 4 m	exactly (by definition)
1 foot = 0.304 8 m	
1 yard = 0.914 4 m	

* Don't worry at this stage about the precise meaning of terms and concepts such as radiation, atoms, wavelength, krypton lamps, etc. We shall be coming back to these ideas in later Units (particularly Unit 9, Units 10 and 11, and TV 12).

** This is perhaps a somewhat unusual meaning of the word 'defined'. It simply means that we have all agreed to *say* that a yard (and hence a foot and an inch) *will be* a certain multiple of the metre.

1.3 Time

The concept of length is relatively straight forward; it is essentially a geometrical concept—a distance between two points in space. The concept of time is not quite so easy. We intuitively think of it as that which occurs between one event and another. But how do we measure time?

To measure length is easy. We can make a metre ruler; we can move the metre ruler from place to place; we can use the metre ruler today and tomorrow and the next day. But an interval of time can be used only once—and then it's gone. Unless, that is, we can find some process which repeats with a regular and countable pattern. You met such a process in Unit 1—the cycle of day following night. Unfortunately, because the Earth's orbit is not perfectly circular, there are slight variations in the Earth's orbital speed during the course of one year. These variations, in turn, cause the interval between successive culminations of the Sun also to vary throughout the year. That is not too good—it means that a 'summer day' and a 'spring day' do not correspond to quite the same time interval. So we must look for a better standard. One possibility is to choose the *mean solar day* as the standard. The mean solar day is the *average* (taken over a year) of the time the Earth takes to spin on its axis once relative to the Sun. The division of this mean solar day into 24 hours, and each hour into 60 minutes, and each minute into 60 seconds then gives us the basic unit of time—*the second.* Time defined in this way is known as *universal time.*

mean solar day

second

universal time

But although this unit of time has been generally successful for most everyday applications, it has again proved unsatisfactory for very high precision work. For, in addition to the variation in the solar day caused by variations in the Earth's orbital speed, there is also a cumulative slowing down of the Earth's spin (probably caused by the effects of tidal friction) which is increasing the length of the *mean* solar day by about fifteen millionths of a second every year. The integrated effect of this is to cause our solar clock to lose over half an hour every 1 000 years.

FIGURE 4 The world's first caesium clock. It is not exactly designed for domestic use, but it does have the virtue of keeping time to better than 1 second in 10 000 years!

So in 1967, a *natural unit of time* was adopted, in much the same way as a natural unit of length had been adopted a few years earlier. And again, this natural unit of time was based upon the identical nature of atoms—only in this case, the *atomic clock* was based on a characteristic vibration of caesium atoms. The *physical second* is now defined as the time required for exactly 9 192 631 770 cycles of this particular vibration in caesium. The world's first caesium clock (Figure 4) was developed at the National Physical Laboratory, England, in 1967. It keeps time to an accuracy of better than 1 second in 10 000 years.

atomic clock

physical second

Current technology, however, is doing even better than this. As you can see from Table 2, 'hydrogen maser' clocks (a *maser* is the microwave equivalent of a *laser*) are beginning to provide a precision of 1 second in 3 million years!

TABLE 2 The precision of different types of clock

Clock	Typical precision
hour glass	1 s in 1.5 min (1 in 10^2)*
pendulum clock	1 s in 3 hours (1 in 10^4)
balance-wheel watch (self-winding)	1 s in 1 day (1 in 10^5)
tuning-fork watch	1 s in 1 week (1 in 10^6)
quartz crystal watch	1 s in 3 years (1 in 10^8)
ammonia resonator	1 s in 30 years (1 in 10^9)
caesium resonator	1 s in 3×10^4 years (1 in 10^{12})
hydrogen maser	1 s in 3×10^6 years (1 in 10^{14})

* This powers-of-ten notation will be discussed in Section 1.6.1.

Needless to say, all this accuracy has brought with it a few attendent problems. On 1 January, 1972, Greenwich mean time changed over to the atomic clock standard. All to the good you might think. But there's a snag. The atomic clock is much more accurate than our old 'solar clock', and the result is that the two clocks are getting steadily out of step. Admittedly, at the moment the discrepancy amounts only to about 0.7 seconds every year, but if this were allowed to accumulate, there would eventually come the day when our atomic clock would be chiming 12.00 midnight at the same time as the Sun, high in the sky, was indicating that it was noon! So, to prevent this happening, we have a 'leap-second', every now and then, during which atomic time is stopped for a full second to give the Earth a chance to catch up.

ITQ 1 If these 'leap-seconds' were not inserted, how many years would it take for an atomic clock to gain 12 hours relative to our 'solar' clock?

1.4 Mass

The concept of mass is fundamentally different from the concepts of length and time; length and time are essentially abstract concepts, associated with space and our perception of events in that space. Consequently, our concepts of length and time do not require any understanding of the properties of *matter*. Mass, on the other hand, *is* a basic property of matter. Indeed, you probably intuitively conceive of mass as a measure of the amount of matter in an object. But if you were to try to work out a theory of the way in which objects move and interact (something we shall be doing in Unit 3), using only this concept of mass, you would soon begin to have difficulties. As you will see in Unit 3, a proper definition of mass requires an understanding of the physics of moving objects.

So how are we to define a *standard of mass*? Well, we choose to adopt an *operational* approach, that is, we agree to call one particular lump of matter (the internationally agreed standard lump is a cylinder of platinum–iridium kept in the International Bureau of Weights and Measures at Sèvres) one *kilogram*. We then compare this standard lump with any other lump of matter by using a beam balance, similar to that shown in Figure 5. When the balance balances, we *define* the mass of the second lump to be the same as that of the standard. Secondary standards of mass made in this way have been distributed to standards laboratories throughout the world. The British *standard kilogram* is shown in Figure 6.

kilogram

standard kilogram

Of course, once a primary standard has been decided upon, it is possible to make a whole range of secondary standards—not just one kilogram, but 2 kg, 5 kg, 10 kg, secondary standards. We simply have to group together a number of (guaranteed) 1 kg secondary standards, and balance them against the required larger standard. We can also make standards that are fractions of a kilogram. For instance, to make a 0.5 kg standard, we simply have to construct

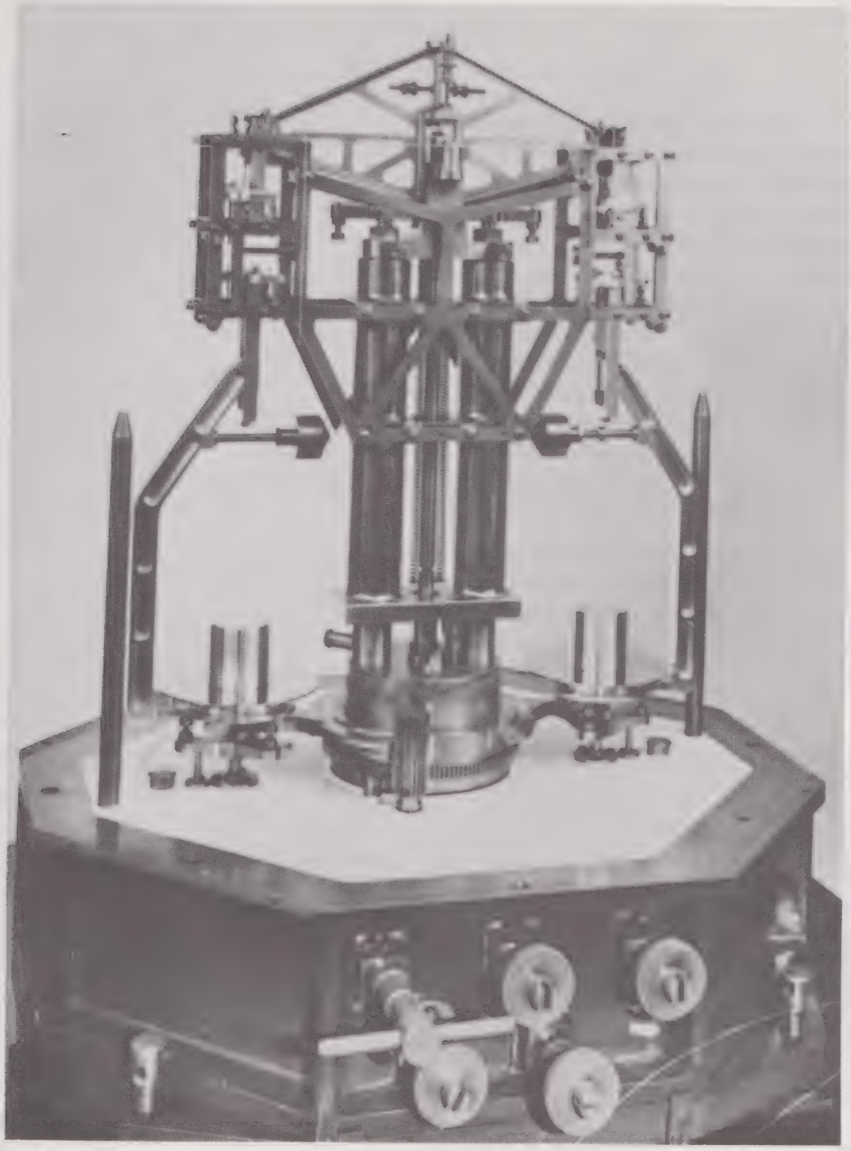


FIGURE 5 The ancient Egyptians (about 5000 B.C.) are known to have used beam balances for comparing masses. The beam balance shown in this photograph is kept at the National Physic Laboratory in Teddington. It is used for comparing other kilogram standards with Kilogram 18 (Figure 6). It is capable of a precision of better than one part in one-hundred million.

FIGURE 6 The British standard kilogram (Kilogram No. 18) is a cylinder of platinum–iridium 39 mm in diameter and 39 mm high.

two equal masses (we test them for equality by balancing them against each other) that together can counterbalance our 1 kg standard.

Naturally, it would be good if we could find an atomic standard of mass to supersede this operational standard. Such a standard does exist for the comparison of the masses of atoms*, but unfortunately we have, as yet, not discovered a way of scaling up this atomic mass standard with sufficient precision to allow us to use it for everyday mass comparisons. But doubtless we shall, one day!

1.5 Length, time and mass—a summary

The basic *unit* of length is the metre. The modern *standard* of length is the wavelength of the orange-red light emitted by krypton gas.

The basic *unit* of time is the second. The modern *standard* of time is a characteristic vibration period exhibited by atoms of caesium.

The basic *unit* of mass is the kilogram. The *standard* of mass cannot yet be defined with sufficient precision in terms of atomic quantities. Therefore, we define the standard of mass, operationally, as the quantity of matter in a particular cylinder of platinum–iridium kept in the International Bureau of Weights and Measures.

* We shall discuss atomic mass units in Units 10 and 11.

1.6 A little more about units

1.6.1 The powers-of-ten notation

Since the choice of our basic units is essentially arbitrary, it would seem sensible to make the size of these units reflect the scale of things, or events, in our everyday experience. The metre, second and kilogram do precisely this. For example, a metre is about the size of a man's stride, a second approximately the time between heartbeats, a kilogram roughly the mass of a large loaf of bread. However, when you consider that science is concerned with the whole range of phenomena in our Universe—from events on the subatomic scale to developments on a galactic scale—you can see that we shall frequently encounter quantities that are either very much smaller, or enormously larger than the size of these basic units. For instance, the distance to our nearest-neighbour star, Alpha Centauri, is about 40 400 000 000 000 000 metres. Or, at the other extreme, the wavelength of yellow sodium light is 0.000 000 589 metres. It is clearly inconvenient to have to write these quantities in this way. Instead, we prefer to use the *powers-of-ten* notation. **powers-of-ten**

The essential idea behind this notation is derived from the observation that several tens multiplied together can generate very large numbers. In the powers of ten notation these numbers can be represented by placing a superscript after the number 10; the superscript indicates the number of tens that have to be multiplied together to get the number. That is

$$10^1 \text{ means } 10 = 10$$
$$10^2 \text{ means } 10 \times 10 = 100$$
$$10^3 \text{ means } 10 \times 10 \times 10 = 1\,000$$
$$10^4 \text{ means } 10 \times 10 \times 10 \times 10 = 10\,000$$
$$10^5 \text{ means } 10 \times 10 \times 10 \times 10 \times 10 = 100\,000$$

We call this superscript the *power* to which ten is raised.

We can cope with small numbers in much the same way. We simply have to take the *reciprocal* of several tens multiplied together. We indicate the fact that we have taken the reciprocal by placing a negative sign in front of the superscript, **MAFS 1***

that is
$$0.1 = \frac{1}{10^1} \text{ is written as } 10^{-1}$$
$$0.01 = \frac{1}{100} = \frac{1}{10^2} \text{ is written as } 10^{-2}$$
$$0.001 = \frac{1}{1\,000} = \frac{1}{10^3} \text{ is written as } 10^{-3}$$

and so on.

With this notation, the number 1 can also be accommodated. Since $10 = 10^1$, and since $0.1 = 1/10 = 10^{-1}$, it makes sense to define

$$1 = 10^0$$

(In fact, we define *any* number raised to the power zero to be 1.)

The powers-of-ten notation can also be used when the number in question is not an exact multiple of ten. For instance, we can write:

$$4\,000 = 4 \times 1\,000 = 4 \times 10^3$$

or similarly:

$$0.06 = 6 \times 1/100 = 6 \times 10^{-2}$$

Consequently, we can now write the distance to Alpha Centauri as 4.04×10^{16} m, and the wavelength of yellow sodium light as 5.89×10^{-7} m. These numbers are much more easily assimilated in this form than they were when they incorporated a large number of zeros.

* We shall use this type of marginal flag throughout the Course to indicate a reference out to *MAFS*, in this case to *MAFS* Block 1.

ITQ 2 Use the powers-of-ten notation to express the number of seconds in one solar day (i.e. 24 hours).

Because of this frequent necessity to use the powers-of-ten notation, scientists have devised a system of abbreviations for some of the powers of ten. However, these abbreviations, which are shown in Table 3, should only be used *as prefixes to a unit of measurement*, e.g. km for kilometres, or ns for nanoseconds. They *cannot* be used alone as abbreviations for 'pure' numbers.

TABLE 3 Prefix abbreviations for various powers-of-ten

Prefix*	Symbol	Power of 10
tera	T	10^{12}
giga	G	10^{9}
mega	M	10^{6}
kilo	k	10^{3}
hecto	h	10^{2}
deca	da	10^{1}
deci	d	10^{-1}
centi	c	10^{-2}
milli	m	10^{-3}
micro	μ	10^{-6}
nano	n	10^{-9}
pico	p	10^{-12}
femto	f	10^{-15}

* Note that when a prefix is placed in front of a unit, in effect, it produces a new unit. Consequently, nm^2 (for instance) should be read as $(nanometres)^2$ and *not* as nano $\times$ $(metres)^2$.

As you can see from Table 3, the prefixes generally change in steps of 10^3 (i.e. 1 000). This is the preferred SI* convention. However, the intermediate prefixes *hecto, deca, deci,* and *centi* are so frequently used that we have also included these in the Table.

ITQ 3 The wavelength of sodium light is 5.89×10^{-7} m. Re-express this wavelength in nanometres (see Table 3).

1.6.2 Orders of magnitude

order of magnitude

We sometimes call a power of 10 an *order of magnitude*. So, for instance, we could say that £1 is an order of magnitude more valuable than a 10p piece, or *two* orders of magnitude more valuable than a 1p piece. However, it is more common for this 'order of magnitude' expression to be used in an *approximate* sense. For example, since the distance from London to Aberdeen is about 490 miles and the distance from London to Milton Keynes about 55 miles, we might say that Aberdeen is an order of magnitude further away from London than is Milton Keynes. For many purposes this kind of statement would be quite accurate enough.

Similarly, in science, it is often very useful to be able to get just a rough idea of the size of some quantity, without having to do an exact calculation or to devise a very careful experiment. In fact, the idea of quoting quantities to 'within an order of magnitude' is so useful, that a special symbol has been devised to represent this kind of relationship. We write: the distance to Alpha Centauri $\sim 10^{16}$ m. The symbol $\sim$ means 'is of the order of'. This statement tells us that the distance involved is 10^{16} m *to within a factor of ten*.

* In 1960, the General Conference of Weights and Measures formally agreed that a system of units called Système International d'Unités (abbreviated to SI Units) should be adopted. It was this Conference that suggested that unit prefixes should be used for increments and decrements of 10^3.

ITQ 4 Calculate, to within an order of magnitude, the number of seconds a person can expect to live.

ITQ 5 You calculated in ITQ 2 that there are 8.64×10^4 seconds in one day. How many seconds are there, *to within an order of magnitude*, in 1 week.

The other symbols that you are likely to come across in this Course are listed in Table 4. You will have already met the familiar 'equals' sign. The $\approx$ sign, meaning 'is approximately equal to' is not quite as loose as the $\sim$ sign.
MAFS 1 It implies the *rounding off** of a quantity, rather than a possible factor of ten uncertainty.

TABLE 4 The meaning of some mathematical symbols

Symbol	Meaning
$=$	is equal to
$\approx$	is approximately equal to
$\sim$	is of the order of magnitude of
$>$	is greater than
$<$	is less than
$\gg$	is much greater than
$\ll$	is much less than
$\gtrsim$	is greater than about...
$\lesssim$	is less than about...
$\geqslant$	is greater than or equal to
$\leqslant$	is less than or equal to

1.6.3 Dimensional analysis

Before leaving this subject of units and standards, there is one final point that is worth mentioning: *all physical quantities must have units associated with them.* After all, one possible definition of what constitutes a 'physical quantity' might be: a physical quantity is any quantity that can, at least in principle, be measured. And since making a measurement implies, as you have seen, comparing the quantity being measured with some agreed standard, it therefore follows that the quantity being measured must take on the *same units* as the standard. This is a point you should take to heart; whenever you write down the numerical value of a measurement, *you must always write down the units as well.*

However, this is not merely good scientific practice; it has its practical uses also. For instance, whenever you multiply two physical quantities together, you must also multiply their respective units. So, 5 metres × 12 metres is 60 metres^2. Or, 3 mm × 2 m is 0.006 m^2. (Make sure that you are happy with this last statement**.) Similarly, whenever you divide two physical quantities, you divide not only the numbers but also their respective units. So, 10 kilometers/5 hours is 2 km hour (i.e. 2 kilometres per hour).

This last statement points out a very remarkable fact. You probably recognize the units of km/hour as the units in which we frequently measure *speeds*. (You would certainly have recognized miles/hour). The fact that a *unit* of speed is a *unit* of length divided by a *unit* of time, follows directly from the relationship:

$$\text{speed} = \frac{\text{distance travelled}}{\text{time taken}}$$

This apparently trivial deduction is, in fact, very far-reaching. You can now see why we don't need to have a *standard of speed* locked away in the Bureau of Weights and Measures. We don't need a standard of speed, because the physical quantity speed can be broken down into the ratio of two other physical quantities, distance and time, for which we do have standards. In fact, if we wished, we could express all physical quantities in terms of the three so-called *dimensions* of length, time and mass†. This would then provide us with a very powerful tool for checking the likely validity of an equation between two physical quantities. For, if we equate two quantities, then the dimensions of both these quantities must be identical—we can only equate like with like. This technique is called *dimensional analysis*.

dimensions

Incidentally, the word *dimensions* is used here, rather than units, to allow for the possibility of equating quantities with units which differ only by a conversion factor. For example, feet, miles and metres are all different units; yet they have a common *dimension*—length. Similarly, hours and seconds, though different *units*, both have the dimension of time. Hence, an equation which balances miles per

* See MAFS, Block 1, if you are unclear about what 'rounding off' means.

** 3 mm = 0.003 m, so 3 mm × 2 m is the same as 0.003 m × 2 m. Alternatively, you could have reasoned that 3 mm = 3×10^{-3} m, so 3 mm × 2 m is the same as 3×10^{-3} m × 2 m = 6×10^{-3} m^2.

† In practice, however, we choose *not* to express some quantities in this way. In SI, *seven* dimensionally independent quantities are recognized. In addition to mass, length and time, there are: electric current (measured in amps), thermodynamic temperature (measured in kelvin), amount of substance (measured in moles), and luminous intensity (measured in candela). You will meet some of these additional quantities in later Units.

hour against metres per second is perfectly valid (provided, of course, that the conversion factor has been incorporated into the question). For although the units do not exactly match, the dimensions do. Both sides of the equation have dimensions of length divided by time*.

So, if at first sight the units in the equation do *not* balance, try breaking down the units into their basic dimensions. These basic dimensions must always balance. If they don't, then something is wrong!

We shall be coming back to this point (with some practical examples) later in this Unit; so don't worry too much if these ideas about dimensional analysis are a little hazy at the moment.

1.7 Objectives of Section 1

Having now studied Section 1, you should be able to:

(a) Handle reciprocals, fractions, and proportions between quantities. (ITQ 1)

(b) Express any number using the powers-of-ten notation (ITQ 2)

(c) Convert physical quantities from one metric equivalent to another, using standard prefixes and powers of ten. (ITQ 3)

(d) Use appropriately the *order-to-magnitude* symbol. (ITQs 4 and 5)

In addition, you should now also:

(e) Understand what is meant by a standard of measurement.

(f) Be aware that all physical quantities must have units associated with them.

(g) Appreciate that any physical quantity can be analysed in terms of its basic dimensions.

(h) Know that this dimensional analysis can be used to check the validity of an equation.

2 The Earth, the Sun and the Moon

2.1 Introduction

We now turn from the theory of measurement to its practice. In Unit 1 you looked at the way in which our present-day model of the solar system was in qualitative agreement with observation. But, of course, the model of the solar system did not develop purely on the basis of qualitative observations. Historically, qualitative observations were mixed in with measurements—measurements that were sometimes accurate and sometimes wildly inaccurate (in terms of our present-day knowledge). And sometimes the inaccurate measurements held back the development of the model. Why were some of the measurements inaccurate? As you might guess, the answer to this question is not simple. In part, the inaccuracies can be attributed to poor tools and observational aids. A lot of the difficulty, however, also lay in the indirect way that the measurements had to be made, or in the rather cavalier way in which the data were extended beyond the range of the actual measurements. Sometimes the reasoning that converted the measurement that was actually made into a numerical value for the quantity that was really wanted, was based on somewhat dubious assumptions.

This, however, is not to devalue the work of our ancestors! Indeed, the tricks they used then are, in many ways, very similar to the tricks we use ourselves today. Whenever the measurement to be made lies outside the scope of current observational techniques, we have to rely on our initiative to devise, and interpret the results of, indirect methods of measurement. And initiative was something the early 'measurers of the solar system' had in abundance. So perhaps we can learn something by looking at the sort of measurements they made, and the sort of reasoning they employed.

* Look again at Table 1 (p. 10) for examples of equations in which the dimensions balance, even though the units do not.

2.2 The size of the Earth

2.2.1 The scientific school at Alexandria

The city of Alexandria was founded, at the mouth of the river Nile, by Alexander the Great during his conquest of Asia minor, Egypt and Persia. It became an important centre of learning during the fourth, third and second centuries B.C., and the Museum of Alexandria (which functioned like an academy and university) attracted some of the foremost Greek scholars of the time. About 300 B.C., a school of astronomy was founded there—a school that was to bring a new attitude to the science. For these astronomers sought to quantify the sizes and distances involved in the solar system and so turn astronomy into a 'real' science. Conjecture was all very well, they argued, but it must be based on measurement.

The obvious starting point was to try to find the size of the Earth itself. How big was the world they lived on? One of the earliest estimates was provided, in about 235 B.C., by the Greek astronomer Eratosthenes. His measuring technique was, of necessity, indirect. After all, in 235 B.C. very little of the world was known to the Greeks. And, of course, he did not have the advantages of our present-day technological aids such as radio communications, or space flight! Yet, starting from the assumption that the Earth was spherical*, he managed to get a value for the circumference (and thus also for the radius) of the Earth that compares very favourably with the accepted present day value. (We believe he was probably within 5 per cent of this value.)

2.2.2 Eratosthenes's method

Here is how Eratosthenes approached the problem. He made the assumption that the Sun was so far away that all sunbeams reaching the Earth were, in effect, virtually parallel. (We shall examine this assumption in detail in the TV programme associated with this Unit, TV 02.) He then simultaneously compared the direction of the vertical at two different locations on the Earth's surface with the direction of the parallel beams of sunlight (Figure 7). But how was Eratosthenes to make *simultaneous* measurements at two different places on the Earth's surface? If the Greeks had had reliable clocks that could be synchronized and then transported, he could have asked an assistant to make the measurement at one of the locations at some previously agreed instant of time (as shown by one of the clocks), while he made the measurement at the other location at the same instant of time (as shown by the second clock). But the Greeks had no such clocks. So Eratosthenes solved the problem of simultaneity by carrying out his measurements at noon (i.e. the time when the Sun was highest in the sky) at two places lying on (what we should now call) the same line of longitude**.

Why did he choose two locations lying on the same line of longitude?

A line of longitude is a line drawn around the Earth in a north–south direction and passing through the two poles. Consequently, all points on this line experience noon (i.e. that time of day when the Earth's rotation brings the Sun to its culmination point) at the same instant of time. The two locations that Eratosthenes chose were Alexandria (where he worked) and Syene—now called Aswan—almost exactly 500 miles due south of Alexandria. It is easy for us to quote this distance nowadays, but Eratosthenes would have found this measurement very difficult to make. We are not sure exactly how he solved the problem—the various documented accounts are ambiguous on this point. Some say that he calculated the distance from the uniform rate of travel of the camel caravans—100 stadia a day for 50 days, giving a total of 5 000 stadia. (The stadium was the unit of length used by Eratosthenes. There is some argument as to how it compares with modern units of length. The Roman writer, Pliny, claims that

* The idea of a spherical Earth was well established in Greek culture. Aristotle had argued that, on the grounds of symmetry alone, the Earth *must* be a sphere. But there was also the experimental evidence provided by (i) the always circular shape of the Earth's shadow thrown onto the Moon during a lunar eclipse, and (ii) the change in the position of the stars as an observer travelled Northward or Southward (see Unit 1). It was not until the Middle Ages that the 'flat earth' became popular again.

** We shall be discussing lines of latitude and longitude in the radio programme associated with this Unit, Radio 01.

Eratosthenes's stadium was of such a size as to give 10 stadia to the Roman mile, where the Roman mile was about 1 per cent shorter than our modern mile. To keep the arithmetic straightforward, we shall assume 1 stadium = 0.1 modern miles, so that 5 000 stadia = 500 miles.) Other sources say that the distance was known from Egyptian survey charts, for which the measurements had been made by professional pacers ('bematists'), who were trained to walk with even strides and count their steps! Whichever technique was used, it was fortunate that the terrain between Alexandria and Syene was sufficiently flat to allow the measurement to be made at all.

But what observations did Eratosthenes actually make at Alexandria and Syene, and how did his observations enable him to estimate the size of the Earth? Well, the observation at Syene was a very simple one. He knew from records kept at Syene that, at exactly noon on midsummer's day, sunlight falling on a very deep well there reached the water surface and was reflected straight back up the well again. (What the records said was that the water in this particular well was only visible at noon on midsummer's day. We would now say that Syene lies on the tropic of Cancer.)

What do you think Eratosthenes deduced from this fact?

Eratosthenes reasoned, therefore, that the direction of the sunlight and the direction of the local vertical at Syene were in coincidence at that particular instant of time. That is, at noon on midsummer's day, the Sun at Syene was exactly 'vertically' overhead. Hence the angle between the direction of the Sun's rays and the direction of the local vertical was zero degrees. (One degree is defined to be 1/360th part of a rotation through a complete circle.)

Consequently, if Erastothenes also measured the angle between the direction of the Sun's rays and the direction of the local vertical at *Alexandria* at noon on midsummer's day, he would, in practice, be measuring the angle between the Earth's radius to Syene and the Earth's radius to Alexandria. These two (equivalent) angles are both labelled θ in Figure 7. (θ, the Greek letter pronounced 'theta', is frequently used to denote angles.)

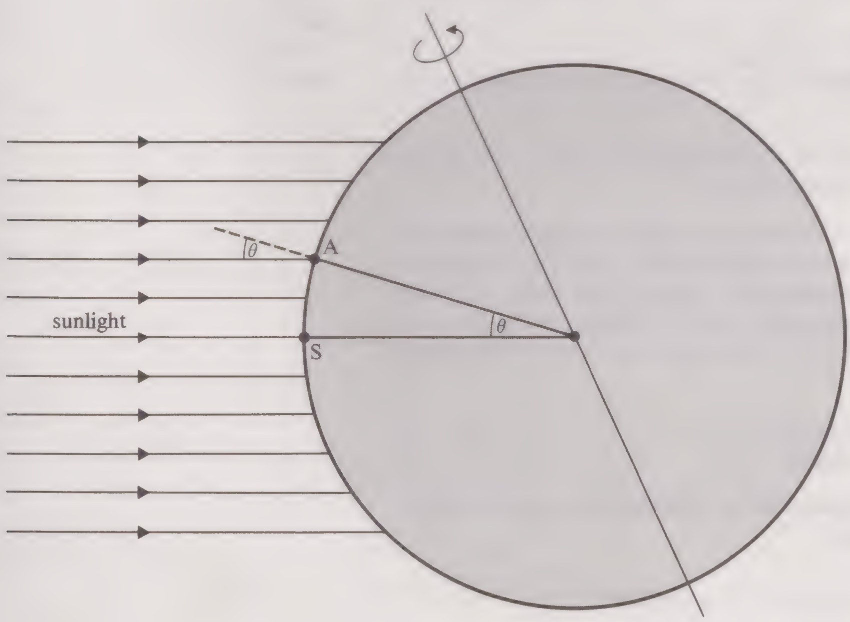


FIGURE 7 The parallel beams of sunlight provide a *reference* direction for all points facing the Sun. The direction of the local vertical (i.e. the direction in which a plumb-line would hang at that locality) is, by definition, the direction determined by a line passing through that point on the Earth's surface, and the centre of the Earth. In other words, the local vertical lies in the same direction as the Earth's radius at that location (plumb-lines point towards the centre of a spherical Earth).

Now, however, he was faced with another problem. It is easy to say: measure the angle between the Sun's rays and the local vertical. Yet how was he to do this in practice?

Can you suggest a way?

The trick is to use the fact that unless the Sun is directly overhead, it casts shadows. So, by placing a vertical pole, whose height he knew, in the ground at Alexandria, and by measuring the length of the shadow cast by this pole at noon on midsummer's day, Eratosthenes was able to deduce that the angle between

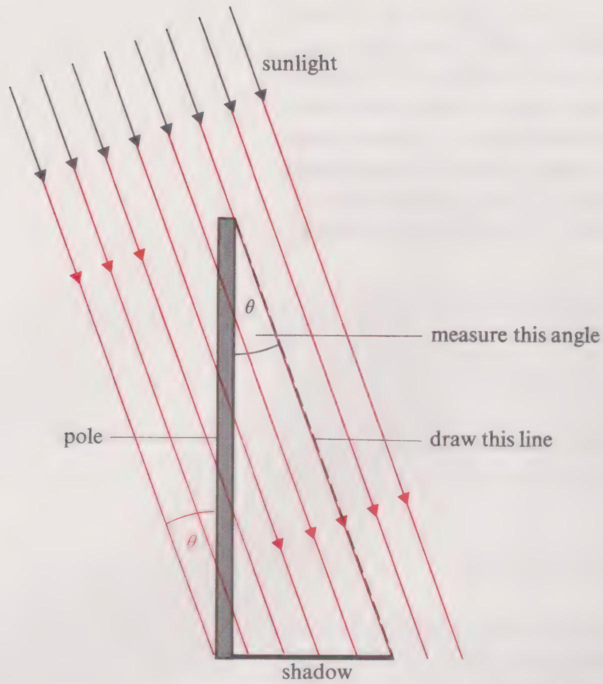


FIGURE 8 If we know the height of the pole and also the length of the shadow cast by it, we can draw a scale diagram (like the one shown here) to enable us to measure the angle between the vertical (the direction of the pole) and the direction of the Sun's rays. (Notice that this angle is exactly equivalent to the angle shown in Figure 7 between the Sun's rays and the vertical, i.e. the angle reproduced here in red.)

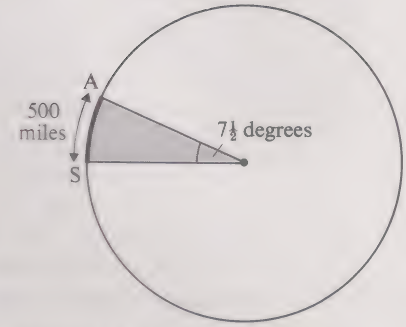


FIGURE 9 An angular separation of 7.5 degrees between Earth radii corresponds to a distance of 500 miles around the circumference of the Earth.

the Sun's rays and the vertical at Alexandria was $7\frac{1}{2}$ degrees (Figure 8). Hence the angle between the two Earth radii—to Alexandria and to Syene—is also $7\frac{1}{2}$ degrees. But Alexandria and Syene are separated by 500 miles. So, if 7.5 degrees between two Earth radii correspond to 500 miles around the Earth's circumference (Figure 9), then:

1 degree between radii would correspond to $\frac{500}{7.5}$ miles around the circumference.

ITQ 6 What distance around the circumference, therefore, would 360 degrees between Earth radii correspond to?

So, in this way, Eratosthenes was able to determine the approximate circumference of the Earth. Finding the radius was then a relatively easy step. The Greeks had long known that, in any circle, no matter how large or how small, the ratio between the circumference of that circle and its diameter was always the same (Figure 10). It is now standard practice to denote this fixed ratio by the Greek letter π (pi). So we write:

$$\left(\frac{\text{circumference}}{\text{diameter}}\right) = \pi \qquad (1)$$

$\pi = \frac{C}{D}$

This can be rewritten to give an expression for the circumference, by multiplying both sides of the equation by *diameter*,

i.e. $$\left(\frac{\text{circumference}}{\text{diameter}}\right) \times \text{diameter} = \pi \times \text{diameter}$$

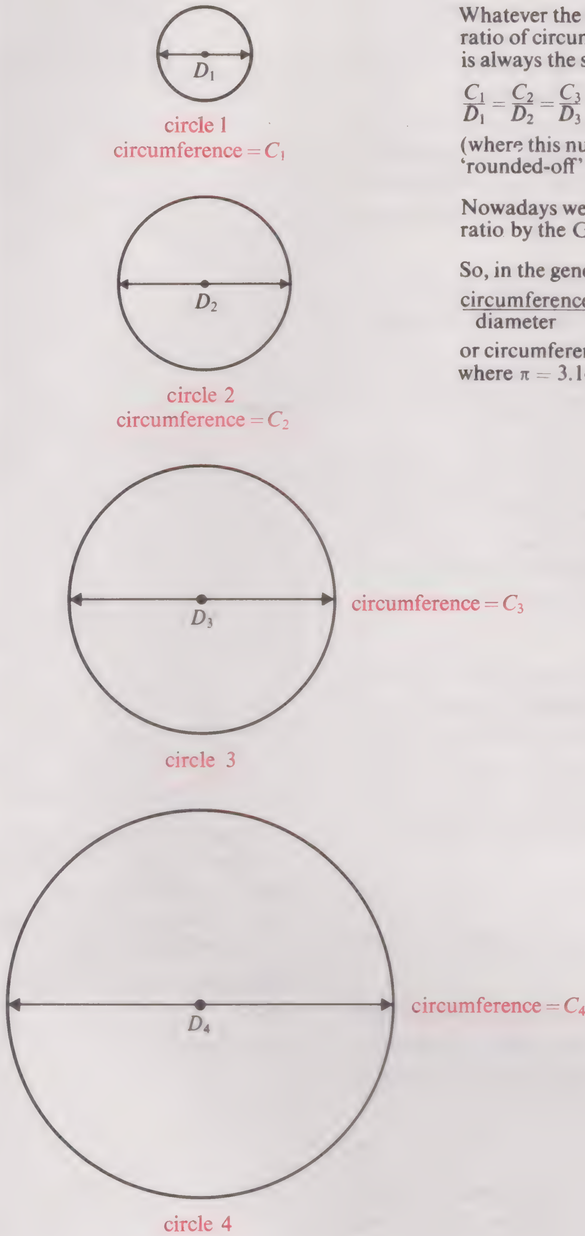
or $$\text{circumference} = \pi \times \text{diameter} \qquad (2)$$

It becomes very tedious having to write out the words circumference and diameter all the time. So it is common practice to represent these quantities by letters. If the circumference is denoted by C, and the diameter by D, then equation 2 becomes

$$C = \pi \times D$$

or $$C = \pi D \qquad (3)$$

(Notice that the multiplication sign between π and D is not necessary. If you see two symbols next to each other like this, the multiplication sign is implied.)



Whatever the size of the circle, the ratio of circumference to diameter is always the same. That is

$$\frac{C_1}{D_1} = \frac{C_2}{D_2} = \frac{C_3}{D_3} = \ldots = 3.14\ldots$$

(where this number has been 'rounded-off' to three figures).

Nowadays we denote this constant ratio by the Greek letter π (pi).

So, in the general case, we say that

$$\frac{\text{circumference}}{\text{diameter}} = \pi$$

or circumference $= \pi \times$ diameter
where $\pi = 3.14\ldots$

FIGURE 10 The definition of π. You will also find π discussed in *MAFS*, Block 4.

The final step is to recall that the diameter of a circle is just twice the radius. So if the radius is represented by the symbol R, then:

$$D = 2R \tag{4}$$

and equation 3 can be written as

$$C = 2\pi R \tag{5}$$

$C = \pi D = 2\pi R$

(where we have simply replaced D by $2R$—the order of multiplication is unimportant)

You probably recognize equation 5. Anyway, the important thing to notice is that if the circumference of the Earth is known, the radius of the Earth can be calculated.

ITQ 7 You calculated Eratosthenes's value for the circumference of the Earth in ITQ 6. If π is taken to be 3.14, what (according to Eratosthenes) is the *radius* of the Earth?

ITQ 8 Given that 1 mile is equivalent to 1.61 km, re-express Eratosthenes's value for the radius of the Earth in kilometres.

2.2.3 An alternative perspective

There is an alternative way of analysing the problem described in the preceding Section, which is well worth taking a look at. We chose not to present this alternative argument from the outset because, to understand it, you need to learn some new mathematical techniques. However, now that you have already seen the basic principles of Eratosthenes's method, and now that you have worked out for yourself the typical radius and circumference of the Earth, you are in a much better position to follow through this alternative approach. Furthermore, the mathematics that you will learn en route will be needed later in this Unit, and will also be used in some of the subsequent Units in this Course. So don't be tempted to skip this Section!

The starting point for this alternative approach is the definition of the units of angular measurement. In the preceding Section, all angles were measured in *degrees*, where one degree was defined to be 1/360 of a complete rotation. However, it is often more convenient in science to use an alternative unit of angular measurement—the *radian*. A radian is defined in the following way.

radian

Draw a circle of radius R. Draw two radii in this circle so that these radii are separated, as in Figure 11, by an angle θ. That part of the circumference of the circle enclosed by the two radii is then known as an *arc* of the circle. Now adjust the angle θ until the length s of this arc (as measured *along* the circumference) is exactly equal to the length of the radius of the circle, that is, adjust θ until $s = R$. When this is the case, we say that $\theta = \theta_R$, where θ_R is *defined* to be *one radian*.

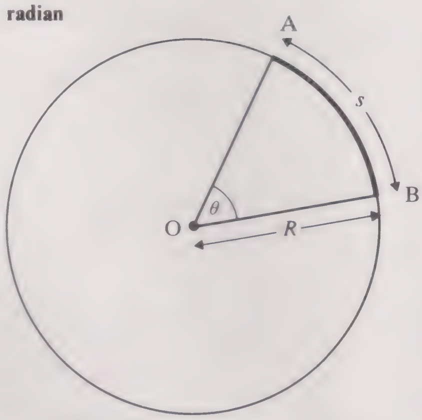


FIGURE 11 The angle θ defines an arc AB of length s. Conversely, we sometimes say that the arc AB *subtends* an angle θ at the centre of the circle. When $s = R$, we say that $\theta = \theta_R$ and define this angle to be 1 radian.

For a general angle of θ radians, $s = R\theta$ (see text).

At first sight this looks to be a very complicated way of defining a unit of angle; but there is method in the madness! First, if an angle of 1 radian means that the arc length is *equal* to the radius, then an angle of 2 radians means that the arc length is equal to *twice* the radius, and an angle of three radians means that the arc length is equal to *three times* the radius, and so on. In general, for a circle of fixed radius R, an increase in θ is going to give a proportionate increase in arc length. The equation describing this relationship is:

$$s = R \times \theta \tag{6}$$

(Convince yourself that this is correct; let $\theta = 1$ radian, and then 2 radians, and then 3 radians)

arc = radius × θ

Equation 6 is an equation you will come across quite frequently—you should memorize it. It says *arc length equals radius multiplied by angle* (*in radians*). But there is something else as well. Recall that the circumference of a circle is equal to twice the radius of that circle multiplied by π,

i.e. $$C = 2\pi R \qquad (5)^*$$

A circumference, however, is simply an arc *that goes all the way round the circle*. So equation 5 is that special form of equation 6, in which the arc length, and hence the angle, corresponds to a complete circle. By comparing equations 5 and 6 we must conclude that the angle corresponding to a complete circle is 2π *radians*. Consequently:

$$360 \text{ degrees} = 2\pi \text{ radians} \tag{7}$$

ITQ 9 A right angle is defined to be 90 degrees. Express a right angle in radians.

ITQ 10 What is 1 radian in degrees?

ITQ 11 Use equation 6 to deduce the *dimensions* of the unit of angular measurement (i.e. radians). Refer back to Section 1 if you've forgotten what is meant by the term 'dimensions'.

How can all this now be applied to Eratosthenes's data? Look at Figure 12. You have already seen that the angle between the direction of the Sun's rays and

* When we requote an equation that has appeared earlier in the text, in this case equation 5, we shall asterisk it.

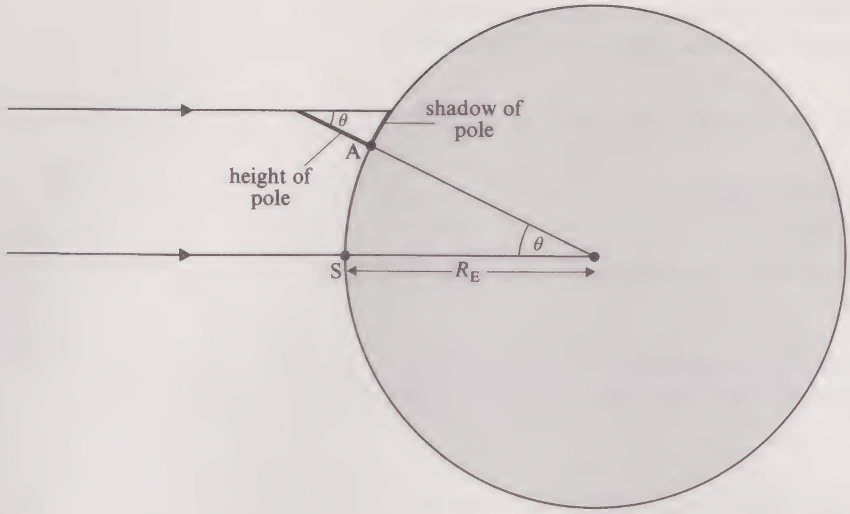


FIGURE 12 A is Alexandria and S is Syene. For clarity, the height and shadow of the pole at Alexandria and the angle θ have all been exaggerated in the diagram.

the vertical direction of the pole at Alexandria was the same as the angle between the two Earth radii to Alexandria and Syene respectively. So if this was $7\frac{1}{2}$ degrees and the distance from Alexandria to Syene was 500 miles ...

> **ITQ 12** Calculate R_E, the radius of the Earth. (The subscript E is used to remind you that the radius in question is the radius of the Earth. This type of subscript notation is very common in science.)

The only thing that is a bit dubious about this last calculation is the value of the angle θ. How was θ actually measured? We suggested earlier that one way would be to measure the height of the vertical pole, and the length of the shadow cast by the pole, and then draw a scaled-down diagram. θ would then have to be measured with a protractor. But this diagram is not really necessary. If we don't mind making a slight approximation in our calculations, we can find the value of θ *directly* from the measurements of shadow length and pole height. We can say that:

$$\text{shadow length} \approx \text{pole height} \times \theta \text{ (in radians)} \qquad (8)$$

Why? Look at Figure 13a. For the circle centred on the tip of the pole (i.e. B), is it not true to say (remembering equation 6) that the arc 'AD' is equal to the radius 'BA' multiplied by θ?* The answer is obviously yes. We are not, however, particularly interested in the length of the arc AD—we are much more concerned with the length of the shadow (AC) along the surface of the Earth. Look at Figure 13b, which is an enlargement of Figure 13a. On this

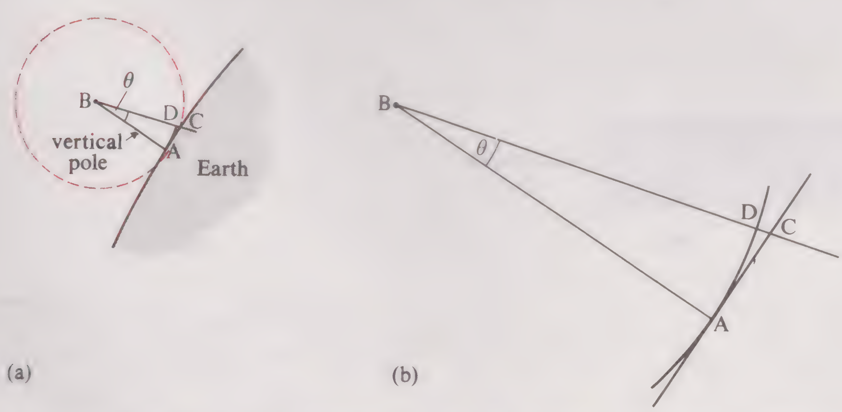


FIGURE 13 (a) and (b) If the angle θ is small enough, the length of the arc AD is approximately the same as the length of the straight line AC.

scale you can see that the shadow AC is, for all practical purposes, a straight line. Furthermore, the length of the arc AD is almost indistinguishable from the shadow length AC. In fact, in order to make the difference between AC and AD visible, the angle θ in this Figure has been made equal to 15 degrees. Eratosthenes's angle was only 7.5 degrees!

* You should be getting used to this notation by now. Radius 'BA' means the radius of the circle whose ends are defined by the points B and A. *It has nothing to do with multiplying a quantity B by a quantity A.* We shall drop the quotation marks around this notation from now on.

So, in summary, if θ is small, the arc AD can be treated as being approximately equal to the length of the shadow along the surface of the Earth. That is why equation 8 is a valid approximation.

If we now rearrange this equation to give an expression for θ, we find that, for small values of the angle,

$$\theta \approx \frac{\text{shadow length}}{\text{pole height}} \tag{9}$$

that is, the value of the angle (in radians) can be determined by calculating the ratio of shadow length to pole height.

> **ITQ 13** You have already seen that Eratosthenes found θ to be 7.5 degrees. Deduce what length of shadow would have been cast at Alexandria by a pole of height 100 length units.

Summary of the main points

1 There are two units of angular measure that are frequently used by scientists: the *degree*, which is 1/360 of a complete rotation; and the *radian*, which is $1/2\pi$ of a complete rotation. They are related by the equation: 360 degrees = 2π radians.

2 The length of an arc of a circle of radius R is related to the angle θ subtended by that arc at the centre of the circle, by the equation:

$$\text{arc} = R \times \theta$$

where θ must be measured in *radians*.

3 When θ is small (anything less than about 15 degrees or 0.25 radians), the curved arc length in this equation can be approximated by a straight line. The smaller the angle, the better the approximation. (The angle of 15 degrees leads to an inexactness of about 1 per cent.) This approximation is sometimes called the *small-angle approximation*.

small-angle approximation

2.3 The size of the Moon

The Earth's nearest neighbour is the Moon. How does the size of the Moon compare with the size of the Earth? How *can* the size of the Moon be compared with the size of the Earth? The early astronomers knew that, when the Earth passed between the Sun and the Moon, a shadow of the Earth was thrown onto the Moon (Figure 14a). Figure 14b shows modern photographs of such a *lunar eclipse*. If it is possible to assume that the shadow of the Earth on the Moon is *the same size as the Earth itself*, then the *ratio* of the size of the Moon to the size of the Earth can be estimated from Figure 14b by completing the circle of the Earth's shadow, and finding the ratio of this circle to the circle of the Moon. Try this yourself. Using a compass* and pencil, complete the circle of the

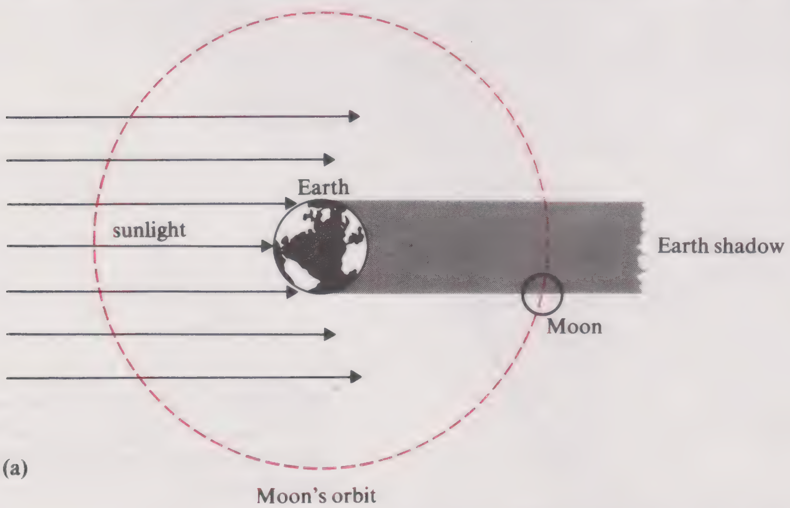


FIGURE 14 (a) The Earth, lying between the Sun and the Moon, casts a shadow on the Moon's surface. (The shadow region shown in this diagram is not *exactly* correct. We shall discuss this in TV 02).

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* If you are at all uncertain about how to use a compass and pencil to draw a circle, you should refer to *MAFS*, Block 4.

FIGURE 14 (b) Photographs showing three partial phases of the lunar eclipse of 2 May, 1920. Notice that the less of the Earth's shadow you have, the more difficult it is to estimate R_E. On the other hand, the more of the Earth's shadow you can see, the more difficult it is to measure R_M.

Earth's shadow. You will probably have some difficulty deciding exactly what size of circle best fits the arc of the shadow. So instead of drawing one circle, draw two. The first circle should be the *biggest* radius circle which you feel *could* be fitted to the arc; the second circle should be the *smallest* circle which you think *could* be fitted to the arc. By doing this you will have estimated, not the exact size of the Earth's shadow, but the *upper and lower limits* to its possible size. You have found the size of the shadow to within certain tolerances. You will find that this is usually what you have to do in science—namely estimate the *range* of possible values that your measurement could cover. No measurement is ever exact—there will always be some *errors or uncertainties* associated with it. So if you can say what the limits of these uncertainties are, other people do at least know what sort of credibility to give your measurement.

upper and lower limits

errors

What values do you get for the upper and lower limits of the radius of the Earth shadow? (Make the measurements in cm.)

The radius of the Earth's shadow in the photograph is:

less than . . . . . . cm

more than . . . . . . cm

ITQ 14 What would you say is the best estimate you can give for the radius of the Earth's shadow?

The next measurement you need is that of the radius of the Moon (as shown in the photograph). Again make this measurement in cm:

The radius of the Moon in the photograph = . . . cm. What tolerance limits do you estimate there are for this measurement?

In this case, you probably felt that you could measure the radius of the Moon (in the photograph) to within say ±0.1 cm. This is a much smaller uncertainty than that involved in the radius of the Earth measurement.

So you can now say what the ratio is between the radius of the Earth and the radius of the Moon:

the Earth is . . . . times bigger than the Moon, that is,
the radius of the Moon = ... Earth radii.

ITQ 15 What are the limits of uncertainty involved in the value for the radius of the Moon?

It is an easy matter now to convert the radius of the Moon (expressed as a fraction of the radius of the Earth) into a measurement in miles.

ITQ 16 Find the approximate radius of the Moon in miles, assuming that the radius of the Earth is about 4000 miles (Eratosthenes's result was probably up to 5 per cent out. So, since we only want approximate values here, it is more convenient to work with the 'round' figure of 4000.)

One word of caution about this value for the radius of the Moon calculated here. You have made an assumption to arrive at this value. You assumed that the shadow of the Earth at the Moon was the same size as the Earth itself. As you will see in TV 02, this is an unjustified assumption that leads to a considerable error in the value obtained for R_M. In TV 02 we shall show you how to make a correction for this error. *Be prepared to modify your value of R_M after watching* this programme.*

2.4 The distance to the Moon

2.4.1 Eclipsing the Moon

So now we know the approximate radii of both the Earth and the Moon. The next measurement we want to make is the *distance between* the Earth and the Moon. Apart from the Sun, the Moon appears as the largest body in the sky. But just because it *appears* to be the largest body, it does not follow that it is the largest body. Indeed, you probably already have a suspicion that the Moon appears larger than the stars because it is nearer than the stars! Apparent size must obviously depend not only on the real size of the object, but also on the distance of the object from

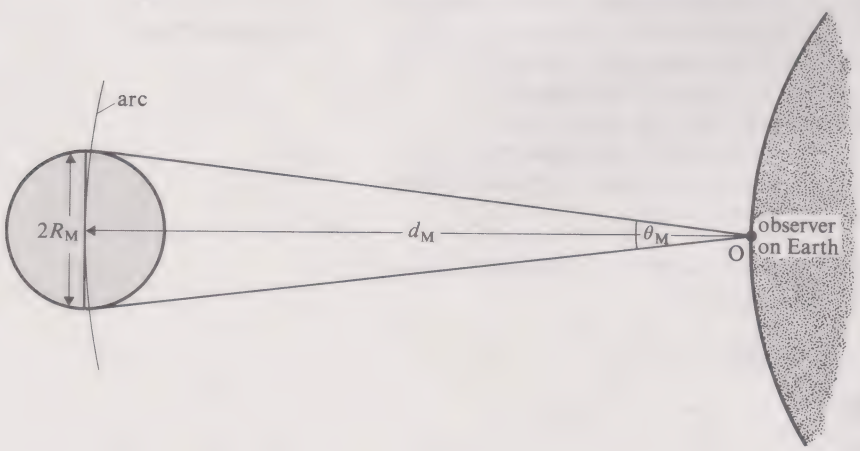

FIGURE 15 If you imagine a circle, centred on O, and passing through the centre of the Moon, the arc corresponding to the angle θ_M will be almost the same length as the diameter of the Moon $2R_M$, provided that θ_M is not too large. Thus, we can write

$$2R_M \approx \text{arc} = d_M \theta_M$$

(The subscript M is used to indicate that we are talking about those particular values of R, d and θ that are relevant to the Moon.)

angular size

you, the observer. The important point to realize is that the apparent size of the Moon is determined by the angle it subtends at your eye**. But what determines the angle? Look at Figure 15. The *angular size* of the Moon in this diagram (i.e. θ_M) is clearly dependent on both the diameter of the Moon ($2R_M$) and the distance of the Moon from the observer on Earth (i.e. on d_M). So, once again, we can use the equation: arc = $r\theta$. The only difference is that here, we are saying:

$$2R_M = d_M \theta_M \qquad (10)$$

* The essential details of this correction are printed in the *Broadcast Notes*.

** Remember that if an arc AB *subtends* an angle θ at a point O, this means that the lines AO and BO are inclined at an angle θ to each other. (Refer back to Figure 11.)

Admittedly, this is an approximation; but if θ is small, the error made by neglecting the curvature of the arc is very small. Remember that, even for θ as large as 15 degrees (about $\frac{1}{4}$ radian), the curved arc and the straight-line approximation to the curved arc only differ by about 1 per cent. In the case of the Moon, θ_M is less than one degree—so equation 10 is a valid approximation.

Rearranging equation 10 to find an expression for θ_M, we have:

$$\theta_M = \frac{2R_M}{d_M} \tag{11}$$

You can see from this, that the angular size of the Moon is determined by the ratio of the Moon's diameter to the Moon's distance from the Earth. You know the Moon's diameter (Section 2.3) so, if you could measure θ_M, you could deduce the distance between the Moon and the Earth. But how do you measure θ_M?

Look at Figure 16. This should give you a clue as to how to measure θ_M. Can you suggest a way?

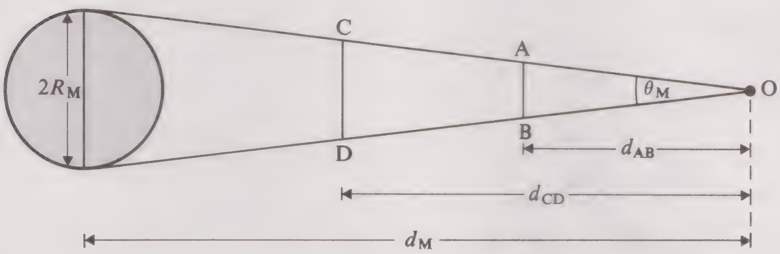

FIGURE 16 The lines AB and CD, and the diameter of the Moon $2R_M$ all subtend the same angle at O, i.e. they all have the same angular size.

θ_M is given by $2R_M/d_M$. But the $arc = r\theta$ equation can also be applied to the 'arc' CD or the 'arc' AB,

i.e.
$$\theta_M = \frac{CD}{d_{CD}} = \frac{AB}{d_{AB}} \tag{12}$$

So, if you were to position some object of known diameter AB (say) a distance d_{AB} away from your eye, such that its *apparent* size (i.e. its angular size θ) was the same as that of the Moon, you could immediately say that θ_M, the angular size of the Moon, is given by AB/d_{AB}. Thus, if d_{AB} can be measured, you can find θ_M (since AB is known).

ITQ 17 Your object of diameter AB, when placed a distance d_{AB} away from your eye, must look about the same apparent size as the (full) Moon. We have already told you that the angular size of the Moon is less than one degree (though we are not saying how much less!). What is the *maximum* diameter object you will need to eclipse the Moon from a distance d_{AB} of 1 metre?

The only question that remains to be answered is: how could you check that your object of diameter AB really is subtending the same angle at your eye as the full Moon? Well, as you'll probably have already guessed by now, the trick is to *just* eclipse the Moon with your object. You then have exactly the arrangement shown in Figure 16. In fact, you don't actually have to work out θ_M in order to find d_M. For, since the equations:

$$\theta_M = \frac{2R_M}{d_M} \tag{11)*}$$

and

$$\theta_M = \frac{AB}{d_{AB}} \tag{12)*}$$

are both true at the same time, we can write:

$$\frac{2R_M}{d_M} = \frac{AB}{d_{AB}} \tag{13}$$

Or, rearranging to find an expression for d_M, we have:

$$d_M = 2R_M \times \frac{d_{AB}}{AB} \tag{14}$$

Note that all the quantities on the right-hand side of this equation are either known or can be measured.

2.4.2 A Home Experiment

You now have all the information you need to enable you to find d_M, the distance between the Earth and the Moon. You should devise for yourself a simple experimental arrangement that enables you to eclipse the Moon. In the first package of your Home Experiment Kit, we have provided you with a selection of small plastic discs, and some Blu-Tack to support them. Perhaps you could use the pieces of dowelling, also provided in the Kit, to act both as a support for the disc, and also as a sighting device, as shown in Figure 17. A tape-measure is also provided in the Kit. However, if you think you can get a more accurate estimate of d_M by improvising your own equipment and techniques—then please feel free to do so.

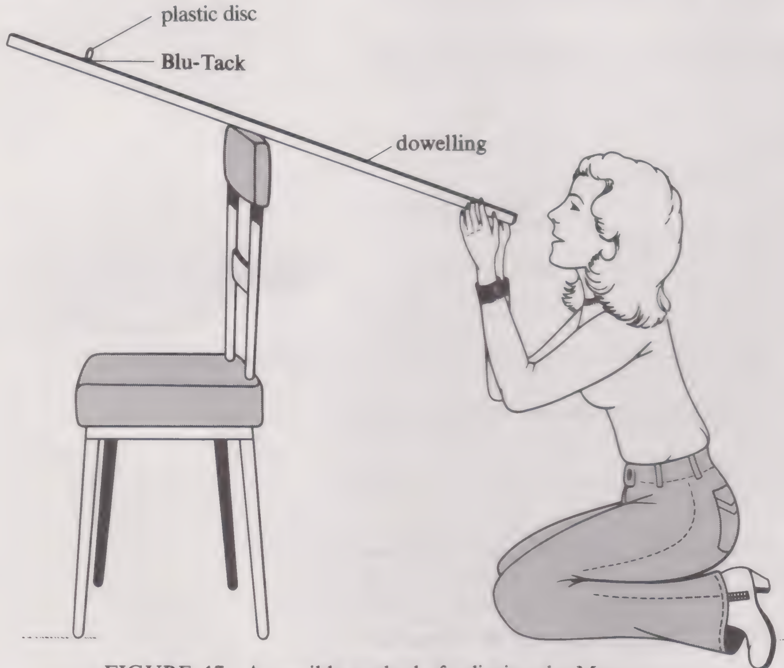

FIGURE 17 A possible method of eclipsing the Moon.

There are a couple of tips that we can offer and that you might find helpful. First, you will find it virtually impossible to block out all the light from the Moon—there will always be some haze around your eclipsing disc. Try instead to match the curvature of the disc to the curvature of the Moon. Second, you will not find it very easy to decide exactly where the optimum eclipse position is. You faced a similar problem to this, in Section 2.3, when you were trying to estimate the radius of the Earth's shadow in the lunar eclipse photograph. There, you found the likely limits, both upper and lower, for this radius. You should do the same sort of thing here; try to estimate the furthest and closest possible eclipse positions, and take the average of these. (You should *always* try to assess the uncertainties (errors) involved whenever you make a measurement.)

You will be asked to 'write-up' this experiment for TMA S101 01, so you should try to make your measurements sometime during this, or next, week. Ideally you require a clear night* and a full Moon to do the experiment properly, but even without these ideal conditions you should still be able to get some kind of result. For instance, if you remember what we said earlier about matching the curvature of the disc to the curvature of the Moon, you should (with a bit of patience) be able to make the measurements on much less than a full Moon. And although there is not much you can do about cloudy nights, you can, at least, practice your eclipsing techniques on household objects (a standard light bulb at 10 metres) indoors, beforehand. You will then be well-prepared to take full advantage of the first available cloud-free night!

One final point. Remember that you will have to revise your value of R_M (the radius of the Moon) after watching TV 02. You should, of course, use this *revised* value in equation 14, to calculate d_M.

* Not quite true! Recall from Unit 1 that the Moon is frequently visible in the daytime.

2.5 The distance to the Sun

One of the first accurate estimates of the Moon's distance from the Earth was made by another early Greek 'astronomer', Aristarchus (about 240 B.C.). And unbelievable though you may find it, using a technique very similar to the one you have just used, he obtained a value which was within a few per cent of our present-day value! When Aristarchus tried his hand at estimating the distance to the *Sun*, however, he got a value which, when compared with the present-day value, was out by a factor of 20 (actually, about twenty times too small)! Why was he so wrong?

The simple answer to this is that the distance to the Sun is much harder to estimate (even today) because the Sun is much further away. But more particularly in Aristarchus's case, the error was so large because he used an indirect technique which, though ingenious, had the defect that a small error in the quantity he *measured*, led to an enormous error in the quantity he was trying to find*. Follow Aristarchus's reasoning through, and see if you can spot where this huge error creeps in.

2.5.1 Aristarchus's value

Aristarchus argued as follows. The Moon goes through various phases—new Moon, half Moon, full Moon, etc.—as the position of the Moon changes relative to the positions of the Earth and the Sun. (Recall Section 3.3 of Unit 1.) But when the Moon shows *exactly* a 'half-Moon' (i.e. first or last quarter), the sunlight must be striking the Moon at right angles (i.e. 90 degrees or $\pi/2$ radians) to the line of sight of the observer watching the Moon. This situation is illustrated in Figure 18. If, at this moment, the observer measures the angle between the direction of the Moon and the direction of the Sun (i.e. ϕ (phi) in Figure 18), he will measure an angle which is almost—but not quite—a right-angle.

FIGURE 18 Aristarchus estimated d_S, the distance to the Sun, by measuring the angle between the Moon and the Sun (the angle labelled ϕ in the diagram) at the moment of time when the Moon appeared to be exactly a 'half-Moon'. He then deduced θ. Since θ was small, he used the equation $d_M = d_S \times \theta$ (which is just arc $= r\theta$) to find d_S. The angle θ has been exaggerated in this diagram to make it clearer. Aristarchus estimated that $\theta \approx 3$ degrees.

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The Greek mathematicians knew, however, that *the sum of all angles in a triangle was equal to 180 degrees or two right-angles*. So, since one of the angles (the angle at M) was a right angle, then

$$\theta + \phi + 90 \text{ degrees} = 180 \text{ degrees}$$

or

$$\theta = (90 - \phi) \text{ degrees} \tag{15}$$

Aristarchus measured ϕ to be about 87 degrees, so he deduced that $\theta \approx 3$ degrees. Now 3 degrees is a small angle, so we can make use of the small-angle approximation for the equation: arc $= r\theta$. In this situation we can write:

$$\text{arc} \approx d_M$$

$$r \approx d_S$$

Therefore

$$d_M = d_S \times \theta$$

or

$$d_S = \frac{d_M}{\theta} \tag{16}$$

ITQ 18 Calculate (using Aristarchus's value of $\theta = 3$ degrees) how many 'Moon-orbit distances' the Earth is away from the Sun. That is, find the ratio d_S/d_M.

* As you should by now appreciate, these are not necessarily one and the same thing. In Section 2.4 you were trying to find d_M. Yet you *measured* the distance d_{AB} and the diameter AB, and assumed R_M.

2.5.2 The present-day value

We now know that Aristarchus's value for ϕ (of 87 degrees) was inaccurate; the present-day value for this angle is 89.85 degrees.

ITQ 19 If we assume that our present-day value of 89.85 degrees for the angle ϕ is accurate to within ± 0.01 degree, it must follow (in retrospect) that the value of 87 degrees obtained by Aristarchus was subject to an uncertainty of at least 2.84 degrees. (Then the upper limit of his value would be consistent with the lower limit of our present-day value.) Thus, the *fractional error* in Aristarchus's value of ϕ must have been about 2.84 parts in 87. This is a *percentage error* of:

fractional error

percentage error

$$\frac{2.84}{87} \times 100 \approx 3 \text{ per cent}$$

So why was his value for d_S out by a factor of 20, when the percentage error in his determination of ϕ was only about 3 per cent?

Try to answer this question before reading on.

So, if θ (i.e. 90 degrees $-\ \phi$) is 0.15 degrees, we can deduce from equation 16 that:

$$d_S = \frac{d_M}{(2\pi \times 0.15/360)} = \frac{d_M}{(\pi/1\,200)} \approx 400 d_M \qquad (17)$$

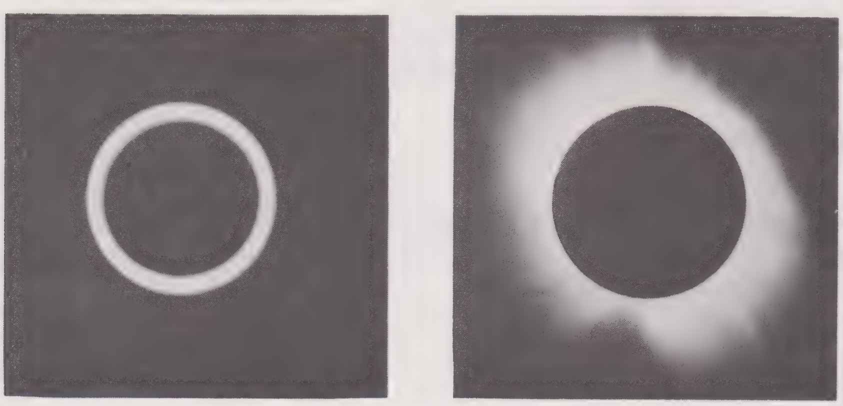
that is, the Sun is about 400 times further away from us than is the Moon. Use your value for the Moon's distance (taken from your Home Experiment*) to calculate the Sun's distance:

$$d_S = \ldots\ldots$$

2.6 The size of the Sun

2.6.1 An interesting observation

How big does the Sun *look* to be in the sky? How big does the Moon *look* in the night sky? Or, more pertinently, how do the apparent sizes of the Sun and Moon compare? Think about this last question for a minute. The interesting observation is that Sun and Moon *look about the same size.* But it is difficult to be sure, because the Sun is so much brighter than the Moon. Is there any independent evidence to support this impression that the Sun and Moon have the same angular size?



(a) (b)

FIGURE 19 (a) A photograph of an *annular* solar eclipse. The Moon does not quite block out all of the Sun.

(b) A photograph of a *total* solar eclipse. The basic disc of the Sun is completely obscured by the Moon so allowing observation of the prominences and flares in the Sun's 'outer atmosphere'.

Actually there is. You've probably seen at sometime or another, photographs of solar eclipses. You've perhaps seen annular eclipses (Figure 19a), or even total eclipses (Figure 19b). Here is direct evidence that the Sun and Moon have approximately the same angular size (Figure 20).

N.B. Under no circumstances should you try to eclipse the Sun with your plastic discs. You will damage your eyes.

* And corrected in the light of TV 02.

FIGURE 20 The Sun and the Moon both have approximately the same angular size—they both subtend an angle of about $\frac{1}{2}$ degree at the Earth. This diagram, which is not to scale, shows how a total eclipse is produced at the point A on the Earth's surface. An annular eclipse (like that shown in Figure 19a) is seen when the Moon is slightly further away from the Earth. The Moon's orbital distance must vary slightly.

2.6.2 The calculation

Figure 20 shows basically the same arrangement as that which you used in the Home Experiment (Section 2.4.2) to find the distance to the Moon. The only difference is that here the Moon is eclipsing the Sun, whereas in your experiment a plastic disc was eclipsing the Moon. Nevertheless, the analysis must be identical, that is

$$\theta_S = \theta_M$$

where
$$\theta_S = \frac{2R_S}{d_S}$$

and
$$\theta_M = \frac{2R_M}{d_M}$$

so that
$$\frac{2R_S}{d_S} = \frac{2R_M}{d_M} \qquad (18)$$

Or, multiplying both sides of this equation by d_S, and dividing both sides by two:

$$R_S = R_M \frac{d_S}{d_M} \qquad (19)$$

Using your value for R_M from Section 2.3 (corrected in the light of TV 02), and assuming that $d_S = 400d_M$, estimate R_S.

$$R_S = \ldots\ldots$$

2.7 Summing up

2.7.1 Tabulating your results

You should now summarize the sizes and distances involved in the Earth–Sun–Moon system by completing Table 5. Use your own data whenever possible in this Table. You should give your results in metres (remembering, if necessary, that 1 mile = 1.61 km). Take care to give your values the correct powers of ten. As your starting point in this Table, you should use the presently accepted value for R_E (the radius of the Earth) of 6.38×10^6 metres. The values you write down should, of course, take account of the correction to R_M discussed in TV 02.

TABLE 5 Summary of results

Measurement	Comment on accuracy
R_E, radius of Earth $= 6.38 \times 10^6$ metres	
R_M, radius of Moon = metres	
R_S, radius of Sun = metres	
d_M, distance from Earth to Moon = metres	
d_S, distance from Earth to Sun = metres	

In the final column of Table 5, you should make some comment on the accuracy of the result you give. If possible, write down error limits; but if this is not possible (because the result is a combination of your own data with data taken from the text, for instance) you should give some idea of the reliability of the result in words.

ITQ 20 How many Earth radii is the Sun away from the Earth?

2.7.2 Completing the picture

All the measurements on the Earth–Sun–Moon system that have been mentioned so far have been *length* measurements; but that is mainly because the timing measurements are relatively easy to make. After all, as you saw in Unit 1, the motion of the Sun relative to the Earth (or vice versa) was used as the basis for the calendar.

For the sake of completeness, however, these time measurements are stated in Table 6.

TABLE 6 Earth–Sun–Moon measurements

Time for Earth to turn once on its own axis (Earth's period of spin)	= 1 day
Time for Moon to orbit around the Earth (Moon's orbital period)	≈ 28 days
Time for Earth to orbit around the Sun (Earth's orbital period)	≈ $365\frac{1}{4}$ days

2.8 Objectives of Section 2

Having completed Section 2 you should be able to:

(a) Calculate the circumference of a circle given the length of arc subtended by a particular angle. (ITQ 6)

(b) Calculate the radius of a circle given its circumference (or vice versa), i.e. apply $C = 2\pi R$. (ITQ 7)

(c) Convert a distance in miles into a distance in kilometres, or vice versa. (More generally, you should now be able to convert from one set of units to another.) (ITQ 8)

(d) Convert an angle from degrees to radians. (ITQ 9)

(e) Convert an angle from radians to degrees. (ITQ 10)

(f) Calculate the dimensions of a quantity that is equal to the quotient (ratio) of two other quantities, when the dimensions of these two quantities are known. (ITQ 11)

(g) Recall that all units of angular measurement are dimensionless quantities. (ITQ 11)

(h) Calculate the length of an arc of a circle given the radius and included angle (or vice versa), i.e. apply the equation: arc = radius × angle. (ITQs 12 and 17)

(i) Use the small angle approximation in the equation: arc = $r\theta$. (Section 2.2.3 and ITQ 18)

(j) Relate the radius of the Earth to the size and shape of the shadow cast by the Earth on the Moon during a lunar eclipse. (Section 2.3)

(k) Estimate upper and lower limits for the value of a measurement. (Section 2.3, ITQ 14 and the Home Experiment)

(l) Calculate a 'best estimate' of a measurement by taking the average of the upper and lower limits of that measurement. (ITQ 14, and the Home Experiment)

(m) Express a measurement as a 'best estimate' together with its upper and lower limits, i.e. in the form $A \pm a$. (ITQ 14, the Home Experiment, and Section 2.7.1)

(n) Express the experimental uncertainty as a fractional or percentage error.

(o) Estimate the upper and lower limits of a quantity that is equal to the quotient (ratio) of two other quantities, when the (known) fractional uncertainty in one of the quantities is much larger than the fractional uncertainty in the other quantity. (ITQ 15)

(p) Devise a way of making a measurement of the distance between the Earth and the Moon, given the diameter of the Moon. (the Home Experiment)

(q) Understand why the percentage error in the *difference* of two nearly equal quantities can be very large, even though the percentage errors in the individual quantities may be small. (ITQ 19)

In addition you should have mastered (with the help, where necessary, of the relevant Sections of *MAFS*) the general objective of being able to:

(r) Handle (i.e. rearrange, or calculate quantities expressed in) mathematical equations involving addition, subtraction, multiplication and division. (ITQs 6–10, 13, 16, 18 and 20)

3 The planets

3.1 Copernicus's contribution

Although the Greek astronomers of Alexandria were able to make quite reasonable estimates of the dimensions of the Earth–Sun–Moon system, they had no real idea of the distances to the planets or the stars. (As you saw in Unit 1, the Greeks knew there was a difference between the planets and the stars; the planets 'wandered' about relative to the star constellations.) The best they could do was presume that the planets were further away than the Moon, and the stars further away than the Sun and planets. It was not until the dawn of the sixteenth century and the rise of the Copernican theory of planetary orbits—with a stationary Sun at the centre of things (Figure 21)—that it became possible to estimate the *relative* distances of the planets. But once the idea had been mooted of a Sun-centred system, with the planets travelling around the Sun in circular orbits, it became possible—admittedly with some fairly complicated reasoning—to begin calculating the relative radii of the orbits.

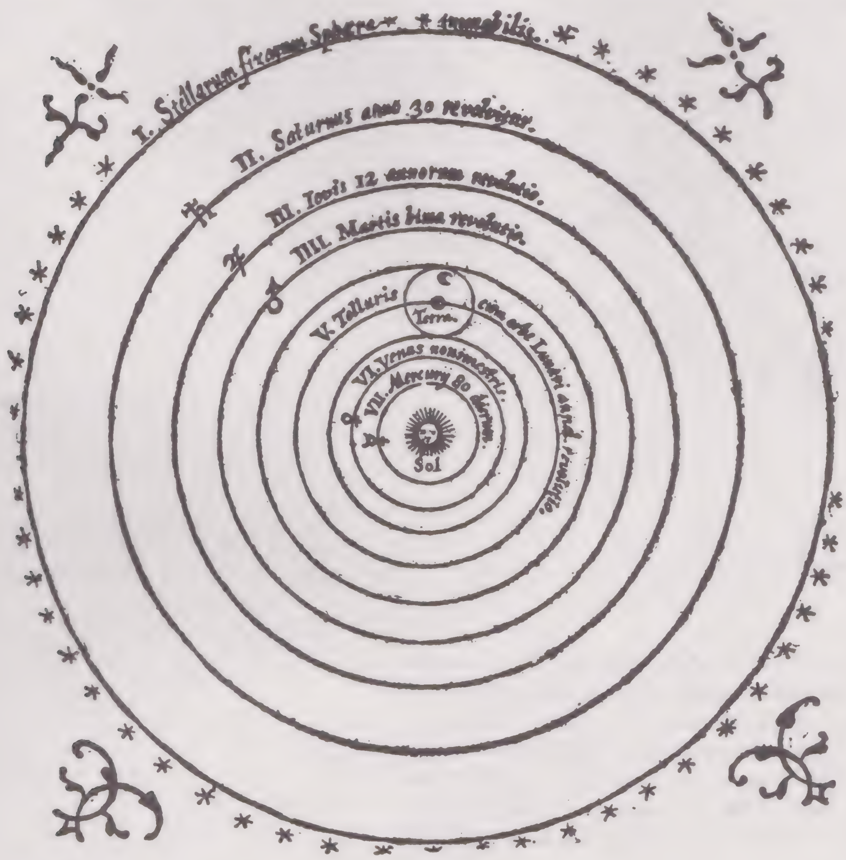


FIGURE 21 The Copernican system of planets with the Sun at its centre. Ironically, Aristarchus had suggested a Sun-centred system back in the third century B.C. Unfortunately, he was ahead of his time—tradition and 'logic' were against him. Furthermore, he was not able to make any measurements to support his 'peculiar' theory.

For instance, Copernicus deduced the ratio of the radius of the Earth's orbit to that of Venus in the following way. (Recall from Unit 1 that Venus's orbit is closer to the Sun than is the Earth's.) Venus's orbit lies in almost the same plane as the

Earth's so that, seen from the Earth, Venus seems merely to swing to and fro, relative to the Sun, first to the left then to the right of it, sometimes passing in front and sometimes behind (Figures 22a and b). As you can see from the two diagrams in Figure 22, the extreme right-hand edge of Venus's apparent oscillation (i.e. the point B) occurs when the line of sight from Earth to Venus *just touches* the circular orbit that Venus is assumed to be following. This line EB, which just touches the circle that is Venus's orbit, is said to be a *tangent* to that circle. You can see that a line of sight to a point earlier in Venus's orbit (e.g. EA), or a line of sight to a point later in her orbit (e.g. EC), always corresponds to a smaller angle of deviation from the Sun's direction. The tangent to the circle corresponds to the angle of maximum deviation of the line of sight. Now the Greek mathematicians had shown long ago that the tangent to a circle is always at right angles to the radius of the circle passing through the tangent's point of contact. So in

tangent (to a circle)

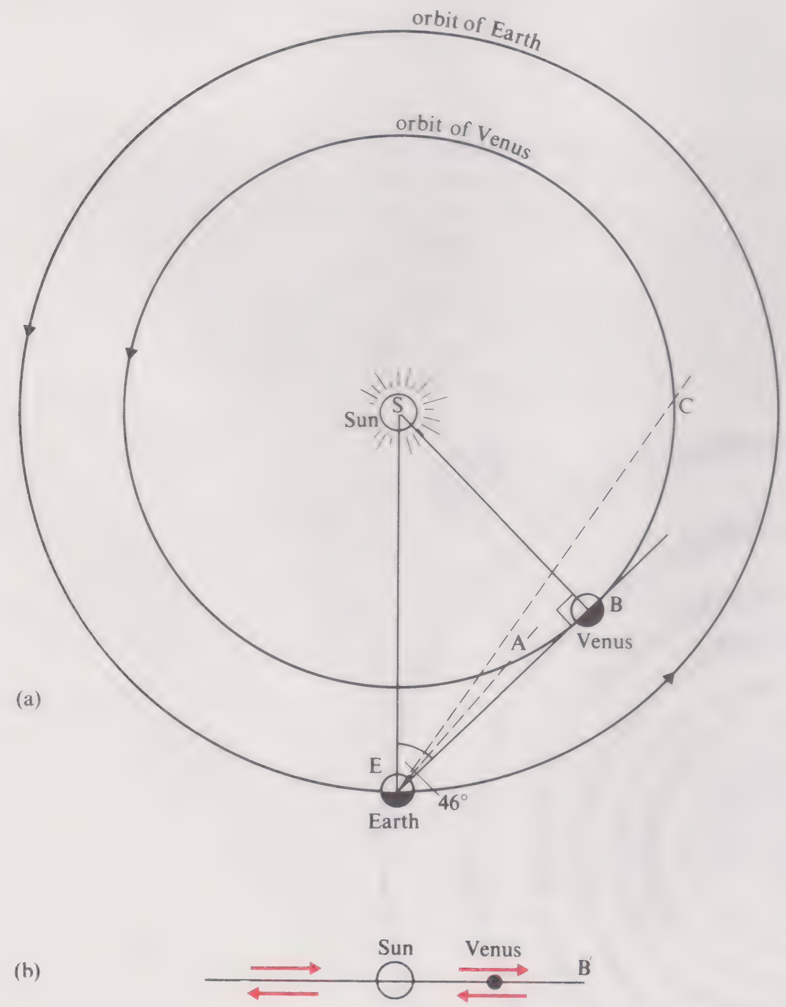


FIGURE 22 (a) The planet Venus orbits about the Sun in more or less the same plane as the Earth. Its orbital radius is smaller than the Earth's.

(b) Venus's orbit, as seen from the Earth, appears to oscillate backwards and forwards across the Sun in a straight line.

Figure 22a we can say that when Venus appears to be at its maximum angular distance from the Sun (i.e. B in Fig. 22b), then the angle at B (we denote it as angle EBS) is 90 degrees. If we also measure the angle at E (i.e. angle SEB) at this instant of time—which we can do by simple sighting from the Earth—then we can deduce the third angle in the triangle BSE. Copernicus found that angle SEB was 46°*. He therefore deduced that the angle ESB (the angle at S) must be 44°. (This follows because 46° + 44° + 90° must add up—as they do here—to 180°.)

Once all the angles of the triangle are known, a scaled down version of this triangle (we call triangles that are identical apart from a scaling factor, *similar triangles*) can be drawn. If the Earth–Sun distance is drawn (arbitrarily) 100 mm long, then it turns out that the Venus–Sun distance must be about 72 mm

similar triangles
MAFS 4

* You will sometimes find units of degrees represented by the superscript circle, as here.

long. (Try it, if you're not convinced.)* Hence the radius of Venus's orbit must be 0.72 (i.e. 72/100) times the radius of the Earth's orbit.

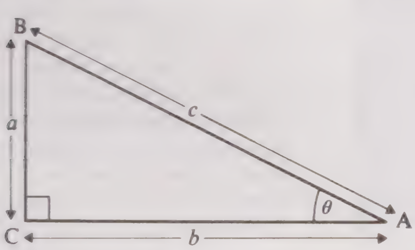
Unfortunately, Copernicus could only turn this *ratio* of orbital radii into an absolute value for the radius of Venus's orbit if he knew the radius of the Earth's orbit accurately (i.e. the value of d_S from Section 2.5). We know this distance quite accurately nowadays, of course, but the best estimate Copernicus had (remember, the telescope had still not been invented in Copernicus's day) was that determined by the early Greeks—and that was a factor of 20 out. Consequently, Copernicus got the *absolute* magnitude of Venus's orbit wrong by this factor.

Needless to say, there was nothing special about Venus. Copernicus repeated similar calculations for the other planets in the system and, although the final *scale* of his solar system was wrong, he did manage to get the *ratios* between the radii of the planetary orbits more or less correct.

Notice, however, that implicit in Copernicus's calculations was the assumption that the orbits were perfectly circular. We now know that this was not a correct assumption. Because of this, the Copernican measurements did not quite fit the facts. For example, since the orbital periods of the planets were quite well known (from the many years' records of detailed sightings), and since Copernicus now claimed to have calculated the relative orbital radii of the planets, it should have been possible to *predict* where a planet would be in the sky (relative to the Earth) at some specified time in the future, or to deduce where it had been some time in the past. It was here that the Copernican model started to show slight discrepancies with the facts. Obviously, his model was still not quite right.

Copernicus spent many years trying to modify his simple model so as to obtain better agreement with the observations. In a way, he succeeded, because he did eventually manage to produce a model in which the agreement between observation and theoretical prediction was very good. But he paid a price for this agreement; his model ceased to be simple! In truth, his modified model of the solar system with the Sun at the centre, was just as complicated as the 'old-fashioned' geocentric (Earth-centred) models. It was perhaps this fact, more than anything else, that gave the champions of the geocentric system—mainly the philosophical and religious bodies of the time—confidence in the 'rightness' of their case.

* You don't *have* to draw this triangle to find the ratio of two of the sides. Instead you can use the mathematical techniques of *trigonometry*. In any *right-angled triangle*, the ratio of any two of the sides can be expressed in terms of one of three quantities. These quantities, which are called sine, cosine and tangent (this tangent is *nothing* to do with the tangent to a circle), are defined by:



$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c}$$ **sin**

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{c}$$ **cos**

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{a}{b}$$ **tan**

where *hypotenuse* is the name given to the side opposite the right angle, *opposite* is the name given to the side opposite the angle whose sine, cosine or tangent is being quoted (angle θ here), and *adjacent* is the name given to the side adjacent to the angle whose sine, cosine or tangent is being found.

These quantities have been painstakingly worked out for all angles between 0° and 90° and tabulated in trigonometrical tables, or programmed into pocket calculators.

So in Figure 22a,

$$\sin 46^\circ = \frac{\mathrm{SB}}{\mathrm{SE}} = \cdots ?$$

(*Answer* 0.72)

If you are not familiar with trigonometry, this subject is further discussed in MAFS, Block 4. You will come across these ideas again in Unit 4.

3.2 Tycho Brahe's tables

The Danish astronomer Tycho Brahe (1546–1601) adopted exactly the opposite approach to that of Copernicus. Rather than invent a model and then try to refine it so as to better fit the facts, he decided to improve the quality and quantity of the facts themselves. So he determined to keep a record of observations of the positions of all the planets (five, plus the Earth, were then known) at regular periods throughout the year. In fact, this collection of data became his life's work; he recorded the planetary positions not for one year, but for more than twenty years. And because of his painstaking development of more and more accurate sighting devices (Figure 23), many of these planetary positions were determined to an accuracy of better than 1/60 of a degree and some to better than 1/360 degree! So Tycho Brahe left to posterity the most accurate and comprehensive catalogue of 'heavenly activities' that history had so far seen. It is no exaggeration to say that these details formed the basis of future developments in the theory of planetary motion. For one of the most challenging tests that any new theory had to pass was that it had to fit the mass of data compiled by Tycho.

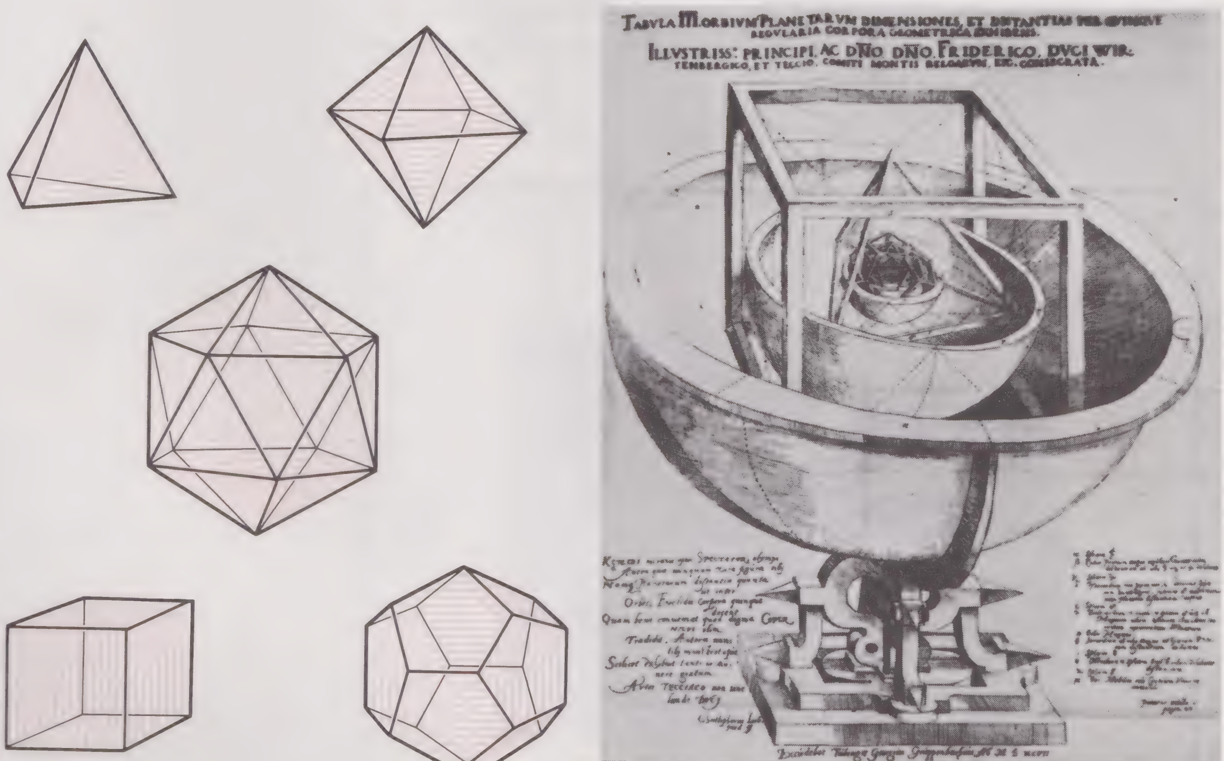
FIGURE 23 One of the most important sighting instruments in Tycho Brahe's observatory at Uraniborg in Denmark was this huge brass quadrant arc. The arc itself was securely fixed into a western wall, and a south-facing open window was located at the centre of the arc. The empty wall-space inside the arc was decorated with a mural showing Brahe, his dog and his laboratories. In this sketch (taken from Brahe's own book), an observer looking through a pin-hole at F (on the extreme right) is locating the position of a 'star ' to an accuracy of better than 1/60 degree (in fact the scale of the instrument was capable of being read to 1/360 degree).

3.3 Kepler's search for regularity

3.3.1 Kepler the 'problem-solver'

Johann Kepler (1571–1630), born in Germany, was a generation younger than Tycho Brahe. As an astronomer, he was everything that Tycho was not. Tycho was a brilliant experimenter and observer. His mechanical ingenuity, as witnessed by his development of numerous observational aids, seemed boundless. Kepler,

on the other hand, was a solver of puzzles. He had the kind of mind that delighted in, and was fascinated by, the relationships between numbers, or sizes, or geometrical shapes. So, in some ways, Kepler was the obvious man to solve the puzzle posed by Tycho Brahe's tables. He also had an almost mystical belief that there was some mathematical scheme underlying the planetary system. Why only six planets? Why did the planets' orbital radii have the ratios 8 : 15 : 20 : 30 : 115 : 195? (These were roughly the relative radii calculated, from Tycho's data, for the Copernican scheme of planets.) Kepler felt sure that there was some sequence to these numbers, and some mathematical explanation for there only being six planets. We know now, of course, that he was wrong. As you will see in Unit 3, the laws of gravitation allow a planet to orbit the Sun at *any* radial distance—so the *ratio* of the orbital radii can have no significance. Furthermore we now know that there are more than six planets—we have added Uranus, Neptune and Pluto to the list. Nevertheless, after months of work, Kepler did come up with an explanation of the ratio of radii—an explanation based on the geometry of the five regular solids (Figure 24). He was so pleased with



(a) (b)

A *regular solid* is the name given to a geometrical solid that has all its faces identical, regular and plane. Hence a regular solid must have all its edges the same length, all its faces the same shape, all its face angles equal, and all its corners identical. Because of the limitations of three-dimensional space, only *five* such solids are possible (see (a)). Kepler had the idea of 'nesting' one of these solids inside a sphere (so that its corners just touched the sphere), and then nesting a second sphere inside the solid (again so that it just touched), and then nesting a second solid inside the second sphere, and so on, until he had used all five regular solids (see (b)). By choosing the correct sequence of regular solids he managed to get close agreement between the ratios of the planet's orbital radii, and the ratios of the radii of the 'nested' spheres. Furthermore, since there are only five regular solids, it is only possible to have *six* 'nested' spheres, thus explaining why there were only six planets! **Kepler was wrong**. We now know that this arrangement between sphere radii and planetary orbital radii *was pure coincidence*. We now also know that there are *nine* planets, not six.

FIGURE 24 The ratios of the orbital radii of the planets—the right answer . . . but the wrong reason. (a) The five regular solids, (b) Kepler's scheme of regular solids (taken from his book).

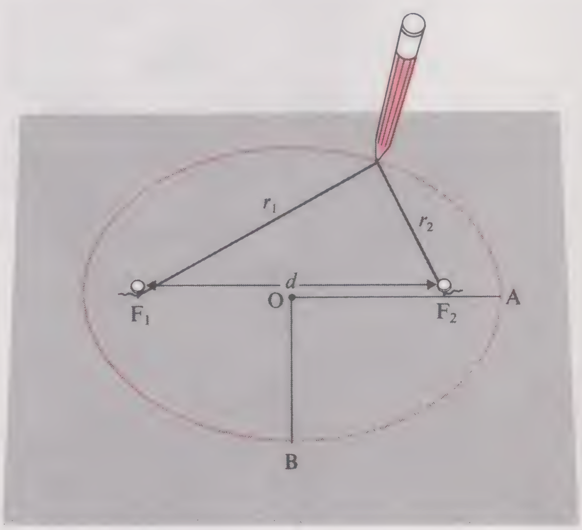
his explanation that he published it in a book, copies of which he sent to Tycho Brahe and the Italian scientist, Galileo. Both scientists were favourably impressed, and Tycho Brahe invited Kepler to go to Prague to work with him on observations of Mars, 'the difficult planet'. So it was that Kepler became acquainted with the detailed tables drawn up over the years by Tycho. Indeed, the tables were still unpublished when Tycho died, and it fell to Kepler to publish them for him posthumously.

3.3.2 Kepler's first law

At the time of Tycho Brahe's death, Kepler was deeply involved in a detailed study of the orbit of Mars. Using the data accumulated by Brahe, he tried to fit Mars first into a circular orbit, and then into a circular orbit with the Sun off centre. Neither worked. Eventually it became clear to him that he would have to plot out, point by point, an accurate and detailed picture of the real orbit of Mars. The problem was not a simple one. All the information was there in Tycho Brahe's tables, but in a scrambled form. The difficulty was that the data gave the apparent position of Mars *as seen from a moving Earth.*

However, Kepler persevered, and after much calculation he determined the *shapes* of the orbits of the Earth and of Mars. The Earth's orbit was very nearly circular; indeed, the apparent deviation from circularity could perhaps have been attributable to experimental uncertainty. But Mars was quite different. The orbital path that he had plotted out for this planet was far from being circular; it was quite clearly 'oval' in shape. Yet, for some reason, Kepler failed for several years to guess that this particular oval shape was an *ellipse*, with the Sun at one focus (Figure 25). This seems strange to us now—we would have thought that the ellipse would be the first and most obvious oval shape to plump for. But then, ellipses are a lot more widely known about today because of Kepler's work on planetary motion!

ellipse



An ellipse is a very easy shape to draw if you have two drawing pins, a piece of string and a pencil. First, fasten the ends of the string to the 'pin-parts' of the two drawing pins. Now press the drawing pins into your drawing surface a distance d apart, where d is less than the length of the string. Take a sharp pencil, and with the tip, extend the string until it is taut. Now draw the curve which the pencil follows when it is moved in such a way as to keep the string taut. This curve is an ellipse; the pins are at the foci of the ellipse. An ellipse can be defined as that curve for which the sum of the distances from the two foci to any point on the curve, is constant, i.e. in the diagram, $r_1 + r_2 =$ constant. The shape of the ellipse can be altered in one of two ways. The distance between the two foci can be changed without changing the length of the string, or the length of the string can be changed without changing the positions of the foci. The distance OA (in the diagram) is known as the semi-major axis, and the distance OB as the semi-minor axis. If the two foci are made coincident (i.e. $d = 0$) the ellipse reduces to a circle.

Kepler's first law says that the path of all the planetary orbits is elliptical, with the Sun at one focus of the ellipse. The other focus has no significance in the case of planetary motion.

FIGURE 25 Defining an ellipse.

Kepler tested out the ellipse idea (with the Sun at one focus) on the other known planets in the solar system. It worked. We know this result as *Kepler's first law.*

KEPLER'S FIRST LAW The planets of the solar system orbit around the Sun along elliptical paths. The Sun lies at one focus of the elliptical orbit.

Kepler's first law

3.3.3 Kepler's second law

Kepler's plot of Mars' orbit, built up as it was from points separated by equal intervals of time, showed him that the planet moved with an uneven speed around its orbit (Figure 26). He was determined to find some pattern behind this 'unevenness'. He had once, much earlier in his life, published a suggestion that a planet was pushed round the Sun by spokes radiating outwards from the Sun—the force behind the spoke being smaller the longer the spoke. The idea sounds ridiculous to modern ears. Planets don't need anything to push them to keep them going. But then, sixteenth-century ideas of motion were rather confused.

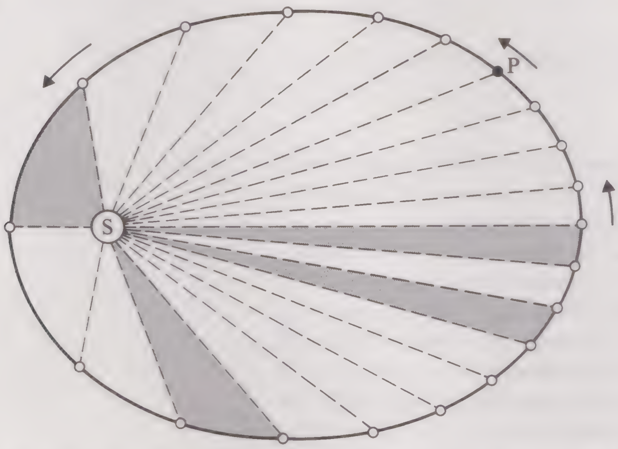


FIGURE 26 The planets move around the Sun in elliptical paths (with the Sun at one focus of the ellipse). The 'spoke' joining Sun and planet sweeps out equal areas in equal times. The planets' positions shown here are separated by equal intervals of time, namely one-twentieth of the planet's 'year'.

Nevertheless, this idea led Kepler to spot a very important relationship. He noticed that the speed of the planet's motion was uneven in such a way as to make the 'spokes' from the Sun sweeps out exactly equal areas of space in equal times. All the shaded areas in Figure 26 are equal. So, if the spoke is a short one (as the planet passes near the Sun), then the speed of the planet must be large to compensate for this. Conversely, the speed is much less when the spoke is long. We now call this result *Kepler's second law*. It works for all the planets.

KEPLER'S SECOND LAW The 'spoke' joining Sun to planet sweeps out equal areas in equal times.

Kepler's second law

Notice that Kepler did not explain *why* this should be the case. He merely observed that it was the case. He had discovered the pattern—but he had not explained it. So, although his second law was a very useful tool for predicting the future positions of planets, it did nothing to explain the 'mysteries of space'. In fact, Kepler's second law is a direct consequence of Newton's (more general) law of gravitation. You will be learning about this law in the next Unit.

3.3.4 The orbital-radius–orbital-period relationship

There was another planetary 'numbers' puzzle that had been worrying Kepler: what was the relationship, if any, between a planet's 'year' (i.e. its time to complete one orbit round the Sun) and its orbital radius? He had all the data from Tycho Brahe's records, but he had not yet spotted the pattern in the data. There was one obvious regularity—the planets with the largest orbital radii* had the longest orbital times. But this is what you would expect—these planets have further to travel in one of their 'years'. Kepler was sure there must be a more definitive relationship hidden away in the data.

* Since the orbits are ellipses, the word radius here should be interpreted to mean 'average radius'. (Strictly speaking, R, as used in Kepler's laws, is the semi-major axis of the ellipse, see Figure 25.)

Table 7 shows the planetary data Kepler had to work on. (Actually, these are modern data, which are slightly more accurate than those used by Kepler.) Can *you* see any relationship between T and R?

TABLE 7 Planetary data

Planet	Orbital radius R (in units of Earth-orbital radii)	Time taken to execute one orbit T (in units of Earth years)
Mercury	0.39	0.24
Venus	0.72	0.62
Earth	1.00	1.00
Mars	1.52	1.88
Jupiter	5.20	11.86
Saturn	9.54	29.46

3.3.5 Plotting a graph

Don't spend too long playing around with the numbers in Table 7—the problem is not trivial. (It took Kepler a long time!) Instead, you should approach this problem in the way that a modern scientist would probably approach it. He would almost certainly plot a *graph**. That is, if a relative orbital radius of 0.39 corresponds to a relative orbital time of 0.24, and a relative orbital radius of 0.72 corresponds to a relative orbital time of 0.62, and so on, then why not show this correspondence on a two-dimensional diagram, in which one quantity is plotted in one direction and the second quantity plotted in a direction at right angles to the first? Then, if there is some fixed relationship between the two quantities, we should expect all the points to lie on a smooth curve; if there is no fixed relationship, we should expect the points to be more or less randomly scattered.

graph

ITQ 21 Can you think of a good argument why a fixed relationship between two quantities should give rise to a smooth curve?

Try drawing this graph now on the graph paper provided in Figure 27**. (The *axes* of the graph have been labelled to help you. Do check, however, that they make sense, because next time you plot a graph you will be on your own!) You will perhaps find it easiest to plot the point corresponding to the largest values of R and T first. Look along the horizontal axis until you find the value 9.54 Earth-orbital radii. Faintly draw in a vertical line at this value. All points on this line must have a value $R = 9.54$ Earth-orbital radii. But you know that this value of R only corresponds to one particular value of T, namely $T = 29.46$ Earth years. So look along the vertical axis until you come to this value of T. Draw in a faint horizontal line at this value of T. All points on this line must have the same value of T. So, it must follow that where your faint vertical and horizontal lines cross, R equals 9.54 Earth-orbital radii, and at the same time T equals 29.46 Earth years. That is, this single point represents both the bits of information corresponding to the planet Saturn.

MAFS 3
axes (of a graph)

Now do the same thing for all the other planets in Table 7. After you've plotted a few points, you'll probably find that you don't have to draw the intersecting lines any more—you'll probably be able to locate their point of intersection by eye. Incidentally, because of the scale of the axes in Figure 27, you'll find that the points corresponding to Mercury and Mars are cramped up near the origin (i.e. the point $R = 0$, $T = 0$). Yet, if we expanded the scale, the point

* It is hard for us to realize nowadays (when graphs are such commonplace devices for displaying data) that the whole concept of graphical representation *post*dated Kepler. In fact, we owe this invention to the mathematician and philosopher, René Descartes (1596–1650). Because of this, you will sometimes hear these graphs referred to as *Cartesian* graphs, or *Cartesian coordinate* systems.

** If you have any difficulty in plotting this graph, refer to *MAFS*, Block 3, for further assistance.

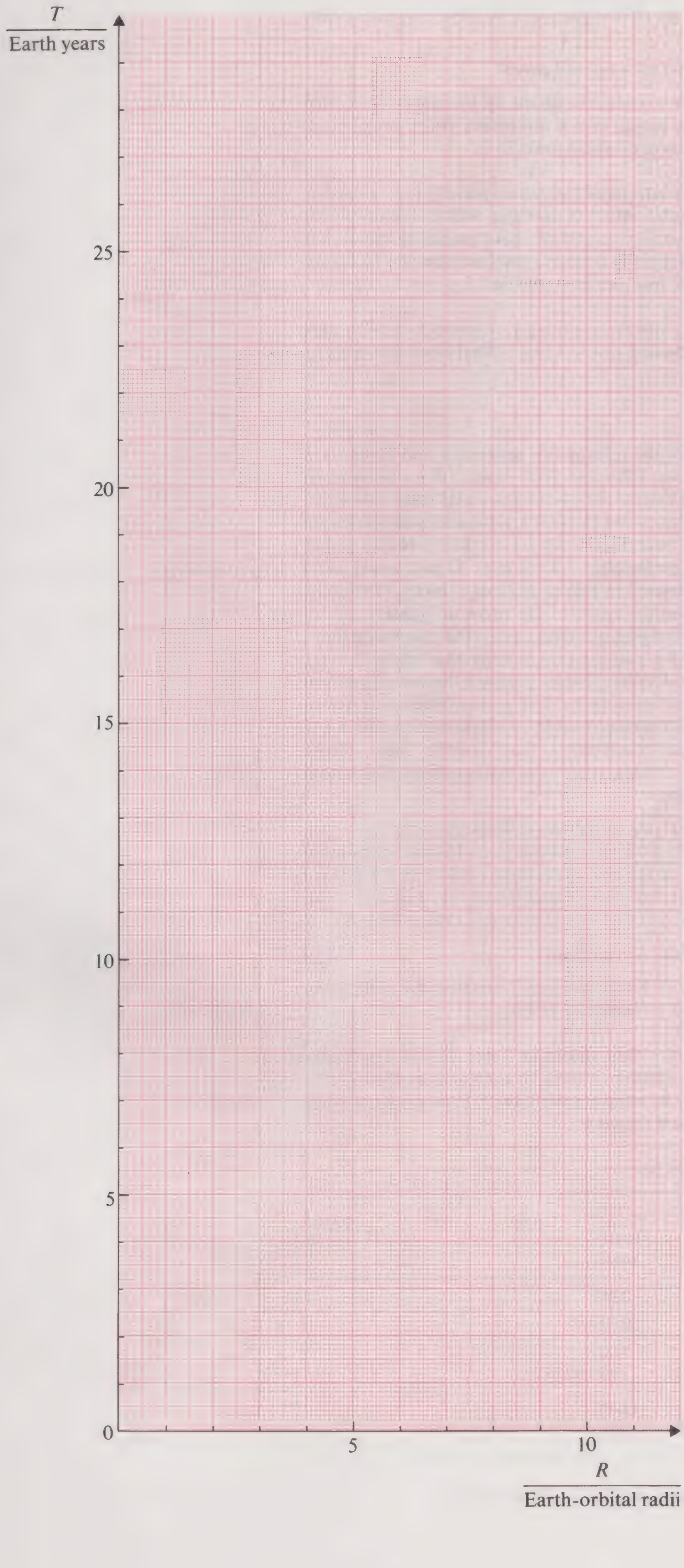


FIGURE 27 Plot points showing the correspondence between R and T for each of the planets given in Table 7. Note that the units of T and R are given below the division line. (This point is covered in more detail in *HED*, see footnote on p. 47.)

corresponding to Saturn would go off the page. So, we have chosen the best compromise.

Do your points appear to lie on a smooth curve?

If they do, draw in this curve. Do *not* join the points up by straight lines—try to estimate the *smoothest* fit to the points, even if this means just missing one or two of them. Your curve should have no sharp kinks in it.

ITQ 22 If you, as a modern astronomer, were unexpectedly to find a planet orbiting the Sun in an orbit with an average orbital radius of 3.0 Earth-orbital radii, would you feel confident about predicting the orbital time period of the planet using only the curve you have sketched in Figure 27? What do you think this time period would be?

ITQ 23 Did you have any trouble in getting your smooth curve to pass through the point corresponding to the Earth? What does this tell you about the Earth?

3.3.6 Deducing the relationship

The graph of Figure 27 does show the relationship between T and R, but in a somewhat 'unspecified' kind of way. Suppose, for instance, that you were to discover a planet with an orbital radius of 39.44 Earth-orbital radii (this is the value of R for the most recently discovered planet, Pluto). Could you deduce the value of T for this planet from your Figure 27 graph? The problem is that you would have to extend the graph (this is known as *extrapolation*) quite considerably beyond the range covered in Figure 27. This is tricky. Obviously, you would do your best to project the shape of the curve to higher values of T and R but, as you can probably imagine, without knowing the 'theoretical form' of the curve (i.e. without knowing the mathematical equation that relates T to R), your projection could be wildly inaccurate. This is precisely the sort of problem that faces social scientists, or governments for that matter, when they try to predict future trends in birthrate, or unemployment, or inflation, etc. without any knowledge of the mathematical relationships that describe these things. Fortunately, with regard to the planets, there *is* a mathematical relationship between T and R. And Kepler found it.

extrapolation

It is asking a bit much to expect you to rediscover Kepler's discovery within your week's work on this Unit! Instead we shall tell you what the relationship is, and ask you to verify that it really does work. Kepler said that if you take the periodic time of a planet's orbit and *square it*, and then divide it by *the cube* of the radius of the planet's orbit, you will always get the same value. That is:

$$T^2/R^3 = \text{a constant} \qquad (20)$$

In fact, if you work in the units of Earth years and Earth-orbital radii (as in Table 7), the constant you get has a magnitude of 1.00.

ITQ 24 Table 8 repeats the data shown in Table 7; but it also has columns for R^3 and T^2. In addition, it has a final column labelled T^2/R^3. Use your pocket calculator to complete this Table*. (You need not work to more than three significant figures.)

MAFS 1

TABLE 8 Testing Kepler's third law

Planet	R / Earth-orbital radii	T / Earth years	R^3	T^2	T^2/R^3
Mercury	0.39	0.24			
Venus	0.72	0.62			
Earth	1.00	1.00			
Mars	1.52	1.88			
Jupiter	5.20	11.86			
Saturn	9.54	29.46			

* For advice on how to use your pocket calculator, see *MAFS*, Block 1.

We now call the relationship expressed in equation 20 *Kepler's third law.*

Kepler's third law

KEPLER'S THIRD LAW The square of a planet's orbital period divided by the cube of that planet's orbital radius is a constant,

or $\frac{T^2}{R^3} = \text{constant}$

In ITQ 24 you found that the constant was 1.00. This, however, was only because the planetary data were expressed in units of Earth years and Earth-orbital radii. Had we used any other units, the constant would not have been unity; *but Kepler's third law (equation 20) would still be true.*

3.4 Uranus, Neptune and Pluto

Since Kepler's day we have discovered three more planets in the solar system—Uranus, Neptune and Pluto. Here is a good opportunity to see if Kepler's third law holds true for these planets as well. After all, Kepler's relationship might not be the only one that satisfies the data relating to the inner six planets—there might be an alternative formula to fit the series of numbers. So a seventh, eighth and ninth planet could be a way of testing the third law.

TABLE 9

planet	R/(Earth-orbital radii)
Uranus	19.14
Neptune	30.20
Pluto	39.44

ITQ 25 Table 9 shows the orbital radii of the three planets Uranus, Neptune and Pluto. What orbital periods does Kepler's third law predict these planets would have?

3.5 Proportionality

Kepler showed that the relationship between T and R is:

$$T^2/R^3 = \text{constant} \qquad (20)^*$$

or, multiplying both sides of this equation by R^3:

$$T^2 = \text{constant} \times R^3 \qquad (21)$$

What this equation says is that any value of T^2 can be found by multiplying the corresponding value of R^3 by a fixed constant, that is, all values of T^2 are *in proportion* to the corresponding values of R^3. So, if R^3 is doubled, T^2 is also doubled; if R^3 is multiplied by four, the corresponding value of T^2 would also be four times bigger.

proportion

In all the cases encountered so far, however, the constant in Kepler's third law has always been 1.00. Thus equation 21 can be reduced to:

$$T^2 = R^3 \qquad (22)$$

But this particular situation exists simply because we have chosen to express T and R in units of Earth years and Earth-orbital radii (both of which are 1.00 for the Earth, of course). With different units, the constant might have been different, say 0.2, or 5 or 7.3. This would not have made any real difference, however; the T^2 values would still have been in proportion to the R^3 values. Only the so-called *constant of proportionality* would have changed.

constant of proportionality

Proportionality relationships occur so frequently in science that we use a special symbol $\propto$ to denote the sentiment 'is proportional to'. For Kepler's law we would say T^2 is proportional to R^3, and write this as

$$T^2 \propto R^3$$

This must be exactly equivalent to the expression which you have already met:

$$T^2 = KR^3 \text{ (where } K \text{ stands for the constant)}$$

This provides you with an important rule: to convert a *proportionality* into an *equality*, a constant—called the *constant of proportionality*—must be included in the equation.

ITQ 26 In Section 1 of this Unit, we stated that the *dimensions* on both sides of an equation must be the same. Kepler's third law says that:

$$T^2 = \text{constant} \times R^3$$

The dimensions of T^2 on the left-hand side of this equation are [time]^2. The dimensions of R^3 are [length]^3. How can this equation be correct?

3.6 Objectives of Section 3

After studying Section 3 you should be able to:

(a) Plot a graph. (Section 3.3.5)

(b) Appreciate that, in general, a graph representing a fixed relationship between two quantities will take on the form of a smooth curve (rather than a random scatter of points). (ITQ 21)

(c) Use a graph (showing the relationship between two quantities) to deduce the value of one quantity given the value of the other. (ITQ 22)

(d) Recognize that, if a set of measurements can be plotted as points on a smooth curve, then any other point which falls on the curve probably represents a measurement within the same generic set. (ITQ 23)

(e) Recall that Kepler's first law states that all planetary orbits around the Sun follow an elliptical path, with the Sun positioned at one of the foci of the ellipse. (Section 3.3.2)

(f) Recall that Kepler's second law states that the imaginary line adjoining the Sun to an orbiting planet sweeps out equal areas in equal times. (Section 3.3.3)

(g) Compile and understand tables of data. (ITQ 24)

(h) Recall that Kepler's third law states that, for any orbiting planet, the ratio of T^2 to R^3 (i.e. orbital period squared to orbital radius cubed) is a constant. (Section 3.3.6)

(i) Use Kepler's third law to calculate the orbital period T (in units of Earth years) of any planet orbiting the Sun, given its orbital radius R (in units of Earth-orbital radii). (ITQ 25)

(j) Convert an expression of proportionality into an equation by introducing a constant of proportionality. (Section 3.5)

(k) Calculate the dimensions of any such constant of proportionality, given the dimensions of the two quantities that are proportional to each other. (ITQ 26)

In addition, you should now also be able (with the help, where necessary, of *MAFS*) to:

(l) Rearrange, and calculate quantities expressed in, mathematical equations involving squares, square roots, and cubes. (ITQ 25)

4 The moons of Jupiter

4.1 Galileo

4.1.1 Galileo's telescope

The other important figure contributing to the science of astronomy at the beginning of the seventeenth century was the Italian, Galileo Galilei (1564–1642). Indeed, it could be said that it was Galileo who laid the foundations of modern observational astronomy by recognizing that the then newly-invented telescope—which made distant objects appear closer, and therefore larger—could be used to advantage in the study of the heavens. His first telescope magnified objects by a factor of only three, but with patience and perseverence he eventually constructed a satisfactory instrument with a magnification of 30 (Figure 28). And with this new instrument he saw, for the first time, the planets not as points of light, but as luminous discs. The 'stars' were still just points in the sky (obviously much further away than the planets) but—through the telescope—they were brighter, further apart, and above all far more numerous.

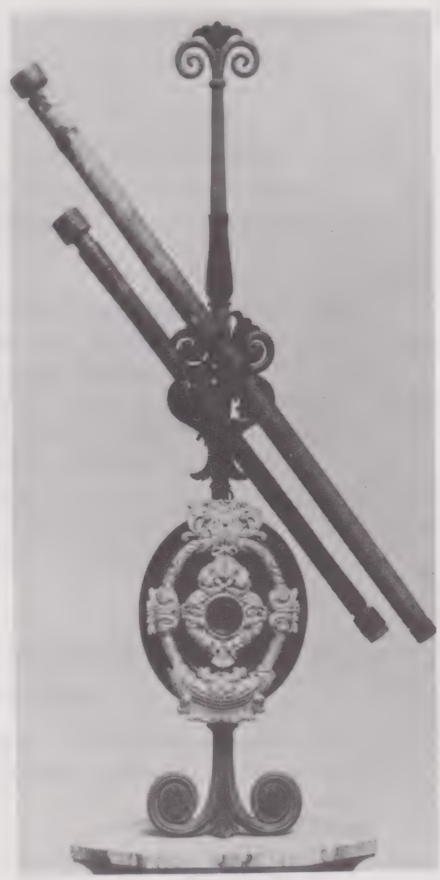
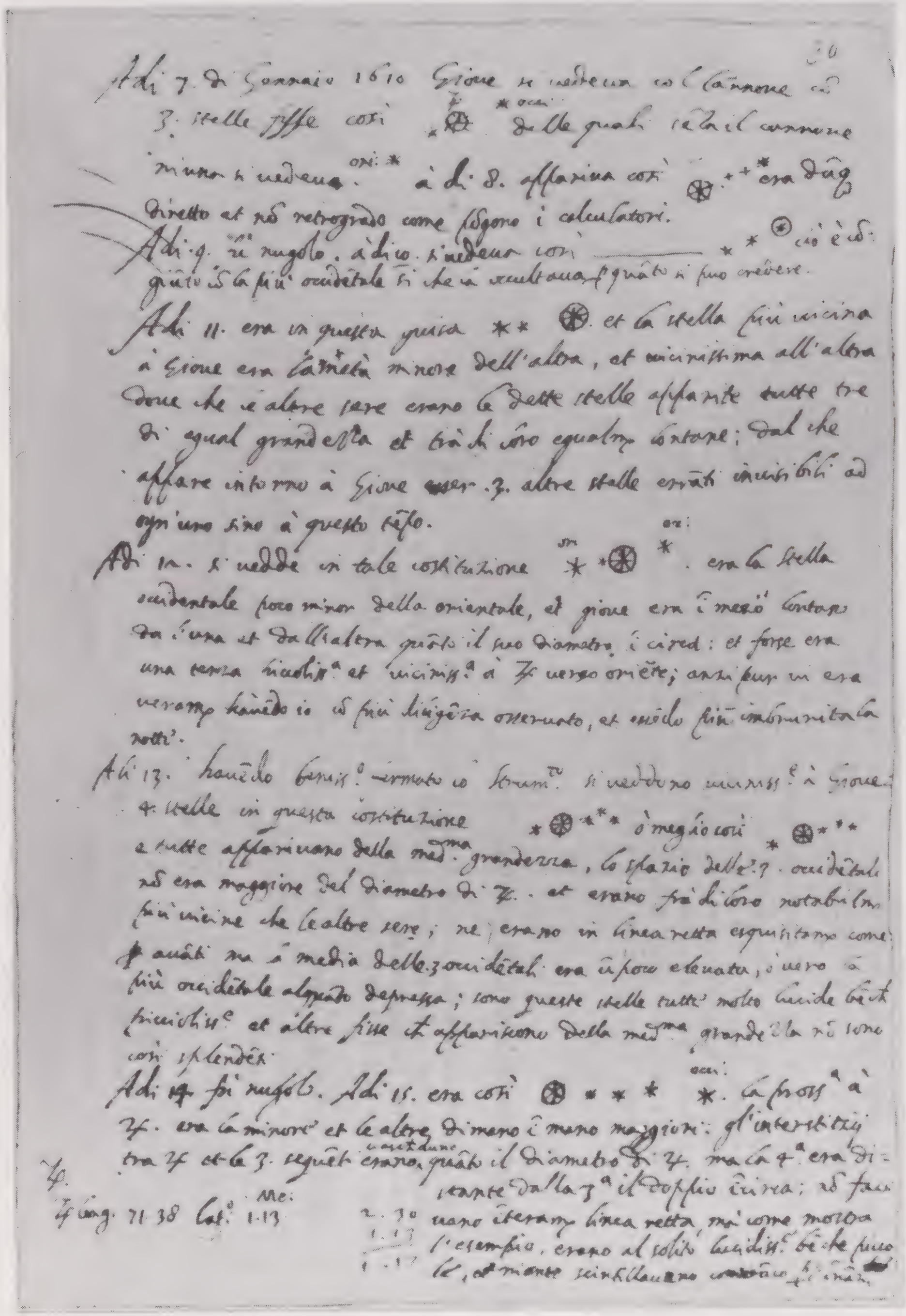


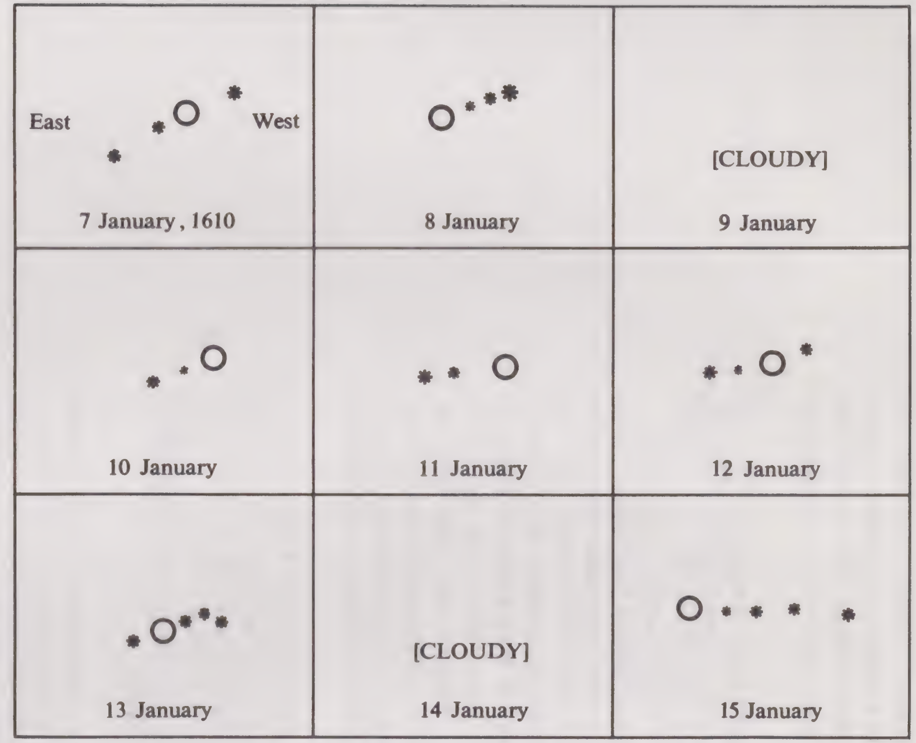
FIGURE 28 Galileo's telescope was probably modelled on the telescope design patented by Hans Lippershey in Holland in 1608.

4.1.2 Galileo's discovery

Perhaps Galileo's most important astronomical discovery was made with this telescope on the night of 7 January 1610. He was studying the region of the sky near the planet Jupiter when he noticed three small new stars. These stars, together with Jupiter itself, seemed to form a straight line; one of the stars was



(a)



(b)

FIGURE 29 (a) A facsimile of the page from Galileo's handwritten notebook showing the positions of Jupiter's moons from 7 January to 15 January 1610.

(b) A summary of the moon's positions each night. The moons' orbits lie almost in the plane containing the line-of-sight from the Earth to Jupiter. Consequently, the moons are frequently in front of, or behind Jupiter; that is why all four moons are not always in view simultaneously.

to the west of Jupiter, the other two to the east. Although he found this straight-line effect sufficiently interesting to make a sketch of the pattern, he did not think there was anything particularly strange about it. He assumed that these stars were simply three more fixed and distant stars that his new telescope had brought within his view.

The surprise came the following night when he was again scanning the sky near Jupiter. The three stars were still there, but now all three were to the west of Jupiter, and positioned more closely together than before. His first thought was that this shift was caused by the motion of Jupiter relative to the Earth—though he did feel that the size of the shift was suspiciously large to have taken place in only 24 hours. But then he realized that, compared with all the other stars, the shift was *in the wrong direction*—Jupiter would have had to be going the wrong way round its orbit! His curiosity was aroused. He decided to watch the 'stars' every night, and keep a record of their positions. Figure 29 shows Galileo's estimate of the position of these 'stars' over the period 7 January to 15 January 1610. The night of 9 January must have been very frustrating for him. The sky was cloudy, and he could not see the 'stars' at all. But, on the tenth, he found the 'stars' had moved back to the east of Jupiter. This convinced him that the stars themselves must be moving, thus indicating that they were not really stars at all.

What Galileo had actually found were four of the moons of Jupiter. He was delighted, and wrote to Kepler about his discovery. Unfortunately, the philosophers of that time were *not* delighted. The discovery was yet more evidence undermining the Earth's unique and all-important position at the centre of the Universe. It is hard now to believe some of the arguments that were advanced against Galileo's discovery. One 'philosopher' wrote:

> There are seven windows in the head, two nostrils, two eyes, two ears and a mouth; so in heaven there are, two favourable stars, two unpropitious, two luminaries, and Mercury alone undecided and indifferent. From which, and many other similar phenomena of nature (such as the seven metals, etc.) which it were tedious to enumerate, we gather that the number of planets is necessarily seven. Moreover, the satellites of Jupiter are invisible to the naked eye, and therefore can have no influence on the Earth, and therefore would be useless, and therefore cannot exist...

Perhaps the worst 'philosophers', however, were those who stubbornly refused to look at the evidence available to them. Galileo wrote to Kepler:

> Oh my dear Kepler, how I wish that we could have one hearty laugh together! Here at Padua, is the principal professor of philosophy whom I have repeatedly and urgently requested to look at the moon and planets through my glass, which he pertinaciously refuses to do. Why are you not here? What shouts of laughter we should have at this glorious folly! And to hear the professor of philosophy at Pisa labouring before the grand duke with logical arguments, as if with magical incantations, to charm the new planets out of the sky

4.2 Jupiter's moons and Kepler's third law

Kepler quickly realized that Jupiter, together with its moons, formed a kind of small-scale model of the solar system. So, if his third law applied to the solar system, why not to Jupiter also? Table 10 shows the orbital radii and orbital periods of the four innermost moons of Jupiter (in metric units this time).

TABLE 10 R and T values for four of Jupiter's moons

moon	R/km	T/hours
Io	4.22×10^5	42.4
Europa	6.71×10^5	85.2
Ganymede	10.71×10^5	171.7
Callisto	18.84×10^5	400.5

ITQ 27 Does Kepler's third law apply to Jupiter's moons?

4.3 Objectives of Section 4

Now that you have read Section 4, you should:

(a) Realize that Kepler's third law applies to *any* 'quasi-planetary' system (though the value of the constant of proportionality depends on what body constitutes the orbital centre of the system). (ITQ 27)

(b) Be able to *verify* that Kepler's third law applies to any such quasi-planetary system, given the orbital periods and radii of the constituent 'bodies' of the system. (ITQ 27)

(c) Be able to evaluate the constant of proportionality of such a system. (ITQ 27)

5 Summary

In this Unit, you have seen the important role that measurement plays in the development of scientific ideas. In particular, you have seen how the 'measurement of the solar system' led Kepler to deduce his three laws of planetary motion—laws which would have been impossible to formulate with only a *qualitative* knowledge of the heavens. It is in this respect that measurement is invaluable in science. We do not want to know the size of things, the distance of things, the duration of things, just for their own sake. We want to be able to use these measurements to help us reject wrong ideas, or to highlight inconsistencies, or to suggest new perspectives. Furthermore, if we can find an underlying pattern to a set of measurements, or deduce a relationship between two sets of measurements (as Kepler did in his third law), then we have taken the first step towards finding a general explanation of the phenomena—towards formulating a theory.

However, in following through the logic of this Unit, you have also, of necessity, achieved something else. You have made a start on learning the language and techniques of science. In some instances these techniques were mathematical—learning to interpret and manipulate equations for example. In other cases, the techniques were experimental—such as estimating limits of uncertainty, or plotting graphs. You will find all these skills listed in greater detail in the following list of Unit Objectives, and you should use this list to check that you really have mastered the most important of them. Remember, however, that learning is rarely a simple 'once and for all' process; you will almost certainly feel a bit unsure about your grasp of some of these Objectives. The important thing is that you have made a start. You will find, as you study the rest of this Course, that these techniques are used again and again; and you will probably find that your mastery of these skills continues to improve as your familiarity with the techniques increases*.

HED

Objectives

After completing this Unit, you should be able to:

1 Handle reciprocals, fractions and proportions between quantities. (ITQs 1 and 6)

2 Use the powers-of-ten notation. (ITQs 2, 3, 4, 5)

3 Explain what is meant by a standard of measurement. (Sections 1.2 to 1.5)

4 Convert physical quantities from one set of units to another, possibly using standard prefixes and/or powers of ten. (ITQs 4 and 5)

5 Use, appropriately, the orders of magnitude symbol. (ITQs 4 and 5)

6 Explain what is meant by the *dimensions* of a physical quantity, and hence calculate the dimensions of such a quantity given an equation relating that quantity to others of known dimensions. (Section 1.6.3, ITQs 11 and 26)

* For ease and convenience of reference, we have collected together in *HED* (The Open University (1979) S101 *The Handling of Experimental Data*, The Open University Press) all the techniques of this kind that you will be learning during your study of this Course (both in later Units, and at Summer School).

7 Calculate the radius (or diameter) of a circle given its circumference (or vice versa), i.e. apply $C = 2\pi R = \pi D$. (ITQ 7)

8 Convert an angle from units of radians to units of degrees (and vice versa). (ITQs 9 and 10)

9 Recall that all units of angular measurement are dimensionless quantities. (ITQ 11)

10 Use the equation: arc = radius × angle ($s = r\theta$) to find one of the quantities given the other two. (ITQs 12 and 17)

11 Use the small-angle approximation in the equation: arc = $r\theta$, (Section 2.2.3) and so relate the *angular size* of an object to its real size and its distance away. (Section 2.4.1, the Home Experiment, and ITQ 18)

12 Relate the radius of the Earth to the size and shape of the shadow cast by the Earth on the Moon during a lunar eclipse. (Section 2.3)

13 Estimate upper and lower limits to a measurement and hence calculate a 'most probable value' by taking the average of the upper and lower limits. (Sections 2.3 and 2.7.1, the Home Experiment, and ITQ 14)

14 Calculate the upper and lower limits of a quantity that is equal to the *ratio* of two other quantities, when the known fractional uncertainty in one of the quantities is much larger than the fractional uncertainty in the other. (ITQ 15)

15 Devise ways of making simple measurements. (the Home Experiment)

16 Understand why the percentage error in the *difference* of two nearly equal quantities can be very large, even though the percentage errors in the individual quantities may be small. (ITQ 19)

17 Compile and understand tables of data. (ITQs 24 and 27)

18 Plot a graph, and (conversely) interpret data plotted in graphical form. (Section 3.3.5, ITQs 21–23)

19 Recall Kepler's laws of planetary motion. (Sections 3.3.2, 3.3.3 and 3.3.6)

20 Recall that Kepler's third law, $T^2 = KR^3$, applies to any 'quasi-planetary' system, and use this law to calculate any one of the quantities T, R or K, given the other two quantities. (ITQs 25 and 27)

21 Convert an expression of proportionality into an expression of equality by introducing a constant of proportionality. (Section 3.5)

There is also a very *general* objective, which has been tested in one form or another in almost all the ITQs in this Unit. It states that:

22 You should be able to rearrange, and calculate quantities expressed in, mathematical equations involving addition, subtraction, multiplication, division, squares, square roots, and cubes.

In addition, there are eight Objectives associated with TV 02 and Radio 01. After watching and listening to these broadcasts, you should be able to:

23 Explain how Eratosthenes's method for measuring the circumference (and thus the radius) of the Earth can be adapted so as not to require the Sun to be directly overhead at either of the two locations (TV 02)

24 Explain why the assumption that the Sun's light forms parallel rays at the surface of the Earth is only approximately true. (TV 02)

25 Estimate the size of the Earth's shadow cast on the Moon in a lunar eclipse. (TV 02)

26 Describe the basic principle on which modern, laser lunar-ranging techniques are based. (TV 02)

27 Appreciate the difference between a rectangular and a polar coordinate system. (Radio 01)

28 Say what angles within the Earth's sphere are defined by the parallels of latitude and the meridians of longitude. (Radio 01)

29 Explain how latitude and longitude may be measured. (Radio 01)

30 Convert angular differences of longitude into local time differences. (Radio 01)

Appendix 1 Some useful numbers

(*Note:* You are not required to memorize any of these data or numbers.)

$\pi = 3.141\,59$ (to six figures)

1 radian = 57.30 degrees (to four figures)

1 degree = 0.017 45 radians (to four figures)

1 inch = 2.54 cm (exactly)

1 cm = 0.393 7 inches (to four figures)

1 mile = 1.609 km (to four figures)

1 km = 0.621 4 miles (to four figures)

Radius of the Earth $\approx 6.38 \times 10^6$ metres (3 960 miles)

Circumference of the Earth $\approx 4.01 \times 10^7$ metres (2.49×10^4 miles)
(distance round the Equator)

Radius of the Moon $\approx 1.74 \times 10^6$ metres (1 080 miles)

Radius of the Sun $\approx 6.96 \times 10^8$ metres (4.33×10^5 miles)

Earth–Sun distance
(i.e. orbital radius of Earth) $\approx 1.50 \times 10^{11}$ metres (9.32×10^7 miles)

Earth–Moon distance
(i.e. orbital radius of Moon) $\approx 3.84 \times 10^8$ metres (2.39×10^5 miles)

ITQ answers and comments

ITQ 1 The atomic clock gains 0.7 seconds in 1 year. It would therefore gain 1 second in (1/0.7) years. But there are $(60 \times 60 \times 12)$ seconds in 12 hours. Therefore, the clock would take $\frac{(60 \times 60 \times 12)}{0.7}$ years to gain 12 hours, i.e. about 60 000 years.

MAFS 1 *If you are not happy with this manipulation of proportions you should refer to MAFS Block 1.*

ITQ 2 There are 60 seconds in a minute, 60 minutes in one hour, and 24 hours in one solar day. Therefore, there are:

$(60 \times 60 \times 24)$ seconds in one day,

i.e. 86 400 seconds in one day

or 8.64×10^4 seconds in one day

ITQ 3 5.89×10^{-7} metres is the same as 589×10^{-9} metres. Thus the wavelength of the sodium light can be written as 589 nm. Alternatively, you may sometimes find this wavelength written as 0.589 μm (i.e. micrometres). This is permissible since 5.89×10^{-7} m is the same as 0.589×10^{-6} m.

ITQ 4 One year is equivalent to: $365\frac{1}{4} \times 24 \times 60 \times 60$ seconds = 31 557 600 seconds.

If we assume a human being lives for about 70 years, then a *typical* lifespan will be 2 209 032 000 seconds. We would write: the lifespan of a human being $\sim 10^9$ seconds.

Notice that it wouldn't have made much difference if you had assumed a lifespan as low as 20 years, or as high as 100 years. Both would have been $\sim 10^9$ seconds.

ITQ 5 Clearly, one week contains $7 \times 8.64 \times 10^4$ seconds = 6.048×10^5 seconds.

To within an order of magnitude, we would say that there are *10^6 seconds* in one week. You should really have been able to write down this order of magnitude answer without having to do the exact multiplication by 7. After all, if you only want the answer to within an order of magnitude, why do the calculation *exactly*? It should have been sufficient for you to say that since 7 multiplied by 8.64 is nearer to 100 than to ten, then 10^4 must be increased by two orders of magnitude to 10^6.

We stated in Table 2 that a tuning-fork watch kept time, typically, to about 1 second in 1 week. You can now see that this statement is the same as saying that it keeps time to 1 second in every 10^6 seconds (to within an order of magnitude). Since this typical precision is unchanged when the units are changed by a simple scaling factor (for example, to 1 hour in 10^6 hours, 1 year in 10^6 years, or 1 *microsecond in every second*), we generally quote the precision as *1 part in 10^6 parts*, or just 1 in 10^6. You should now go back to Table 2 and convince yourself that the two ways in which the typical precision figures are quoted there are equivalent—to within an order of magnitude.

ITQ 6 360 degrees would correspond to $\frac{360 \times 500}{7.5}$ miles = 24 000 miles around the circumference. But remember that we define an angle of 360 degrees to be the angle of a complete circle. So the circumference distance corresponding to 360 degrees must be the *complete circumference*. Hence the circumference of the Earth is *24 000 miles*.

ITQ 7 $C = 2\pi R$ (5)*

Therefore $R = \frac{C}{2\pi}$

Hence, putting in values for C and π,

$$R = \frac{24\,000}{2 \times 3.14} \approx 3\,820 \text{ miles (when rounded off to three figures)}$$

If you are not happy with the re-arranging of equation 5 employed in this ITQ, you should refer to MAFS, Block 2. MAFS 2

ITQ 8 Eratosthenes's value for the radius of the Earth was 3 820 miles. 1 mile is 1.61 km. Therefore 3 820 miles is $(3\,820 \times 1.61)$ km, i.e. the radius of the Earth is about 6 150 km (according to Eratosthenes). You may be interested to know that nowadays the generally accepted value for the radius of the Earth is about 6 380 km. The difference between this value and Eratosthenes's value is 6 380 km − 6 150 km = 230 km. As a fraction of the accepted value, this difference is 230 km/6 380 km $\approx$ 0.036. Expressed as a *percentage*, this difference is $0.036 \times 100 = 3.6$ per cent. We stated (Section 2.2.1) that Eratosthenes's value for the radius of the Earth was probably 'within 5 per cent' of the accepted present day value. The 3.6 per cent calculated here is probably a little too generous to Eratosthenes—after all, we are not quite sure, to within about another 1 per cent or so, exactly how his stadia should be converted into present-day units of length.

ITQ 9 A right angle is 90 degrees; but 90 degrees is one-quarter of 360 degrees. Hence, in radians, a right angle must be equal to one quarter of 2π radians:

i.e. 90 degrees = $2\pi/4$ radians

= $\pi/2$ radians

ITQ 10 2π radians = 360 degrees

Therefore
$$1 \text{ radian} = \frac{360}{2\pi} \text{ degrees}$$
$$\approx \frac{360}{2 \times 3.14} \approx 57.3 \text{ degrees}$$

ITQ 11 Equation 6 says that:

arc = radius × angle (in radians)

or angle = arc/radius (23)

The arc length has dimensions of [length]; the radius also has dimensions of [length]. Hence, the dimensions on the right-hand side of equation 23 must be [length]/[length]. That is, the right-hand side of this equation is *dimensionless*! If the dimensions on both sides of the equation are to balance, then the units of angle must also be dimensionless.

A radian is a dimensionless unit.

The same is true of any other unit of angular measure—the degree, for example, is also dimensionless. This is because an angle is really only a way of expressing a *fraction* of a rotation. 3 degrees really means 3/360 of a complete rotation; 2 radians really means $2/2\pi$ of a complete rotation.

ITQ 12 The arc 'AS' (i.e. the distance along the circumference from A to S) is given by:

$$\text{arc 'AS'} = R_E \times \theta \qquad \text{(from equation 6)}$$

or equivalently $\quad R_E \times \theta = \text{arc 'AS'}$

Dividing both sides of this equation by θ gives an expression for R_E,

i.e. $$R_E = (\text{arc 'AS'})/\theta \qquad (24)$$

But θ must be in radians. So,

since $$360 \text{ degrees} = 2\pi \text{ radians}$$

$$1 \text{ degree} = \frac{2\pi}{360} \text{ radians}$$

and $$\theta = 7.5 \text{ degrees} = \frac{7.5 \times 2\pi}{360} \text{ radians}$$

You don't need to work this out yet. Instead, just replace θ (in equation 24) *by this complete fraction*—after all, they are equal.

That is $$R_E = (\text{arc 'AS'})/\theta \qquad (24)^*$$

$$= (500 \text{ miles}) \Big/ \left(\frac{7.5 \times 2\pi}{360} \text{ radians}\right)$$

$$= \frac{500 \times 360}{7.5 \times 2\pi} \text{ miles}$$

$\approx$ 3 820 miles (when rounded off to three figures)

This is exactly the same answer that you arrived at before.

MAFS 1 *If you are not happy with the way in which this fraction was rearranged, try some of the examples in MAFS, Block 1.*

ITQ 13 Equation 9 says that:

$$\theta \text{ (in radians)} \approx \frac{\text{shadow length}}{\text{pole height}}$$

for small angles. Eratosthenes's value of θ was 7.5 degrees, or $\frac{7.5 \times 2\pi}{360}$ radians (see ITQ 12).

Substituting this value of θ into equation 9, together with a pole height of 100 length units, gives:

$$\left(\frac{7.5 \times 2\pi}{360}\right) \text{radians} \approx \frac{\text{shadow length}}{100 \text{ length units}}$$

Multiplying both sides of this equation by 100 length units gives:

$$\left(\frac{7.5 \times 2\pi}{360}\right) \text{radians} \times 100 \text{ length units} = \text{shadow length.}$$

That is

$$\text{shadow length} \approx 13.1 \text{ length units.}$$

ITQ 14 Probably the best thing to do in this particular case is to take the *average* of the upper and lower limit value. That is, the best estimate is:

$$R_E \text{ (shadow)} = \frac{(R_E)_{max} + (R_E)_{min}}{2}$$

For example, *if* you estimated that the largest possible radius was 12.5 cm, and the smallest possible radius was 7.5 cm, you should express your result as

$$R_E \text{ (shadow)} = 10.0 \text{ cm} \pm 2.5 \text{ cm}$$

where adding on the 2.5 cm gives the upper limit, and subtracting the 2.5 cm gives the lower limit.

(You should read the equation as: R_E equals 10.0 cm, plus or minus 2.5 cm.) *You will find more information about taking aver-*

HED *ages in The Handling of Experimental Data.*

ITQ 15 Suppose you found the radius of the Moon in the photograph to be 4.0 cm ± 0.1 cm*, and the radius of the Earth 10.0 cm ± 2.5 cm. Then the *average* ratio of Earth radius to Moon radius is:

$$\frac{10.0 \text{ cm}}{4.0 \text{ cm}} = 2.5 \text{ times}$$

The *largest* value for the ratio would be obtained if you were to use the *upper* limit for the Earth radius and the *lower* limit for the Moon radius,

i.e. $$\text{maximum possible value of } R_E/R_M = \frac{12.5}{3.9}$$

$$= 3.2 \text{ times}$$

Conversely, the *smallest* value for the ratio is obtained by using the lower limit for R_E and the upper limit for R_M,

i.e. $$\text{minimum possible value of } R_E/R_M = \frac{7.5}{4.1}$$

$$= 1.8 \text{ times}$$

So we can write:

$$R_E = (2.5 \pm 0.7)R_M$$

Perhaps you would like to check that you would have got more or less the same result if you had used 4.0 cm for R_M in both cases (rather than using 3.9 cm and 4.1 cm respectively). The reason why the uncertainty in R_M has very little effect on the final ratio is that the fractional (or percentage) uncertainty in R_M is swamped by the ten times larger fractional (or percentage) uncertainty in R_E.

This point is worth remembering. Whenever you combine several results, all of which have possible uncertainties associated with them, then the percentage error in the combined result will always be larger than the percentage error in any of the individual results. But if the percentage error in one measurement is much bigger than the percentage error in any of the other measurements—the latter uncertainties can usually be neglected. *This point is further discussed in HED.* HED

ITQ 16 In the imaginary example worked out in ITQs 14 and 15, the Moon was approximately two-fifths the size of the Earth,

i.e. $$R_M \approx \frac{2 \times 4\,000}{5} \text{ miles} \approx 1\,600 \text{ miles}$$

You should, of course, calculate the radius of the Moon using your own estimate of the ratio between the two radii. You should also estimate the upper and lower limits for this radius. How important is the possible 5 per cent error in your assumed value of R_E?

ITQ 17 Remember that $\theta_M = AB/d_{AB}$ only if θ_M is measured in radians. Recall that 1 degree = $2\pi/360$ radians. If d_{AB} = 1 metre, then:

$$AB = d_{AB} \times \theta_M = \frac{1 \times 2\pi}{360} \text{ metres}$$

$$\approx 1.7 \times 10^{-2} \text{ metres}$$

Hence, the diameter of the object needed to eclipse the Moon from a distance of 1 metre is *less* than 1.7 cm, i.e. less than the size of a half-penny ($\frac{1}{2}$p) coin.

* These are 'made-up' figures—not what you should actually get.

ITQ 18 Remember, for arc = $r\theta$ to work, θ must be in radians.

$$3 \text{ degrees} = \frac{2\pi \times 3}{360} = \frac{\pi}{60} \text{ radians}$$

Therefore, from equation 16:

$$d_S = \frac{d_M}{(\pi/60)} \approx 20 d_M$$

i.e. the Sun is twenty times further away than the Moon (according to Aristarchus). We now know that this ratio is about 20 times too small!

ITQ 19 The reason for the discrepancy lies in the fact that Aristarchus did not use ϕ to find d_S, but rather θ, which is (90 degrees $- \phi$). Thus he used a value θ of 3 degrees when he should have used a value of 0.15 degrees. *This value of θ really is a factor of 20 too large*, and it therefore causes (through equation 16) d_S to be a factor of 20 too small.

There's a moral here. Whenever a quantity is calculated by taking the *difference* between two measured quantities *that are very nearly equal*, then there is always the possibility of there being very large percentage errors in the calculated quantity, even though the percentage errors in the measured quantities may be quite small. Furthermore, the more nearly equal the two quantities to be subtracted are, the greater the percentage error in their difference is likely to be. So beware this kind of situation!

ITQ 20 If you have obtained sensible values for the distances and sizes asked for in Table 5, then you should find that;

$$d_S \approx 24\,000 \text{ Earth radii}$$

If you obtained a value for d_S between $18\,000 R_E$ and $30\,000 R_E$, you've done pretty well! Mind you, if you got a value *very* close to $24\,000 R_E$, then you have either been very lucky, or you've been cheating!

ITQ 21 This is a difficult question to answer satisfactorily. Perhaps the best clue is contained in the words 'a fixed relationship'.

If there is a 'fixed relationship' between R and T (to take Kepler's problem as an example), then there is always only *one* way of calculating T if you know R (and vice versa). For example, *suppose* the 'fixed relationship' were $T = 2R$. Then for any value of R, *no matter how big or how small R might be*, T would always be found by multiplying R by two.

Now consider one specific value of R and its corresponding value of T. Suppose R is increased by a small amount. Obviously T will increase by a *correspondingly* small amount (related to R through the fixed relationship $T = 2R$). Successive increases in R will produce corresponding increases in T, that is, as R changes *smoothly*, T also changes *smoothly*. So whenever there is a *fixed relationship* between two quantities, the curve showing this relationship will, in general, be a *smooth* curve.

No such restriction need apply if there is *no* fixed relationship between the two quantities. In this situation, when R is changed by a small amount, it is quite possible for T to take on *any value at all*. Consequently the points could be randomly scattered across the graph paper.

ITQ 22 If you obtained a smooth curve for Figure 27 (and you should have done), you should be feeling fairly confident that there is some form of fixed relationship between R and T. If this is the case, then you can deduce pairs of values for R and T other than those corresponding to the six planets you have plotted.

In other words, you would be able to say that, if a planet had an orbital radius R around the Sun of 3.0 Earth-orbital radii, then it must also have a period T of about 5.2 Earth years. How do you know that it is 5.2? Simply by reading from your graph that T value corresponding to an R value of 3.0. That is, read vertically up the graph at $R = 3.0$ Earth-orbital radii until you get to the curve, and then note the T value corresponding to this point on the curve. This is shown in Figure 30. This process of finding pairs of values on a graph (here, $T = 5.2$ at $R = 3.0$) lying *between* two plotted values, is known as *interpolation*.

ITQ 23 The curve you have drawn should pass easily through the point corresponding to the Earth, which fits nicely between the points corresponding to Venus and Mars. What this tells you is that Earth is the same kind of 'animal' as the other planets. Copernicus was right. There is nothing special about the Earth. The relationship between T and R, which you have found in graphical form in Figure 27, is a relationship which works for all 'planets' circling the Sun—including the Earth!

ITQ 24 The square of a number is the number that results when the original number is multiplied by itself. So, $(0.24)^2 = 0.24 \times 0.24 = 0.0576$. The cube of a number is the number which is obtained when the original number is multiplied by itself, and then multiplied by itself again. So, $(0.39)^3 = 0.39 \times 0.39 \times 0.39 = 0.0593$. You should work out T^2 and R^3 in the same way for all the remaining planets.

If you are uncertain about powers of numbers you should read MAFS, Block 1, and try the examples there. **MAFS 1**

As you can see from the final column of Table 11, T^2 divided by R^3 comes to almost exactly unity for all the planets. The slight deviation is never more than a few per cent (a few parts in one hundred) and can be attributed to experimental error*.

* Error used in this context does *not* mean mistake. It means experimental uncertainty.

TABLE 11

	R (Earth-orbital radii)	T (Earth years)	R^3 (Earth-orbital radii)^3	T^2 (Earth years)^2	T^2/R^3 (Earth years)^2/(Earth-orbital radii)^3
Mercury	0.39	0.24	5.93×10^{-2}	5.76×10^{-2}	0.97
Venus	0.72	0.62	3.73×10^{-1}	3.84×10^{-1}	1.03
Earth	1.00	1.00	1.00	1.00	1.00
Mars	1.52	1.88	3.51	3.53	1.01
Jupiter	5.20	11.86	1.41×10^2	1.41×10^2	1.00
Saturn	9.54	29.46	8.68×10^2	8.68×10^2	1.00

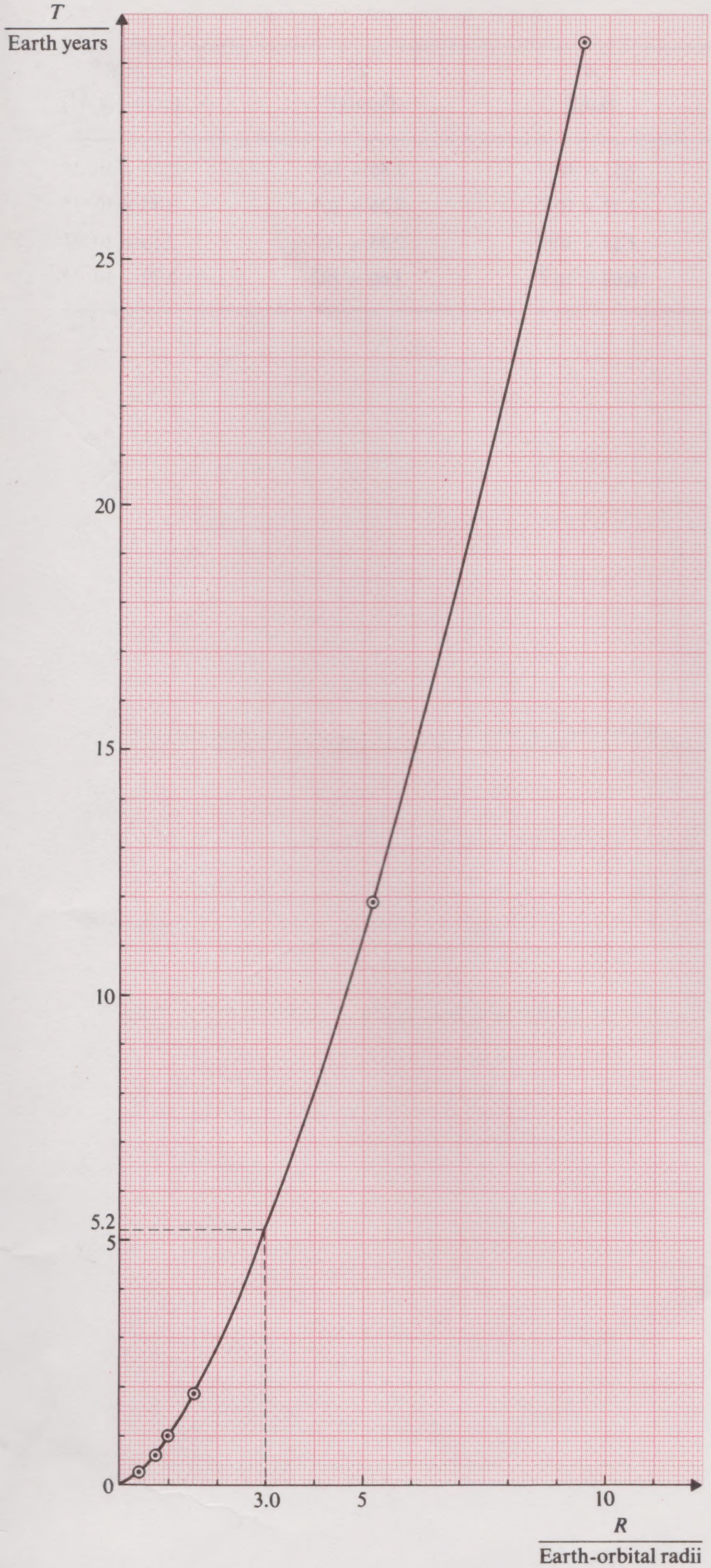


FIGURE 30 The graph of orbital period T against orbital radius R. When $R = 3.0$ Earth-orbital radii, $T = 5.2$ Earth years.

ITQ 25 Kepler's third law says that:

$$\frac{T^2}{R^3} = \text{constant} \qquad (20)^*$$

If T is measured in Earth years, and R in Earth-orbital radii, then the constant is exactly 1.00. Hence

$$T^2 = R^3 \qquad (22)^*$$

so

$$T = \sqrt{R^3} \qquad (25)$$

Alternatively, using the fractional index notation, we can write

$$T = (R^3)^{\frac{1}{2}} \qquad (26)$$

(A number to the power one-half means the square root of the number; a number to the power one-third means the cube root of the number; a number to the power $1/n$ means the nth root of the number.)

If you are not happy with this notation, you should see MAFS, Block 1. **MAFS**

Using the radius values of Table 9 in equation 26 gives:

Uranus: $T = (19.14^3)^{\frac{1}{2}}$

$\approx (7\,011.7)^{\frac{1}{2}}$

≈ 83.7 Earth years

(*Calculator keying sequence:* 19.14, y^x, 3, =, $\sqrt{x}$)

Neptune: $T = (30.20^3)^{\frac{1}{2}}$

$\approx (27\,543)^{\frac{1}{2}}$

≈ 166.0 Earth years

(*Calculator keying sequence:* 30.2, y^x, 3, =, $\sqrt{x}$)

Pluto: $T = (39.44^3)^{\frac{1}{2}}$

$\approx (61\,349)^{\frac{1}{2}}$

≈ 247.7 Earth years

(*Calculator keying sequence:* 39.44, y^x, 3, =, $\sqrt{x}$)

These are the values of T *predicted* by Kepler's third law. The modern measurements for these values of T are 83.74, 165.95 and 247.69 Earth years, respectively. It looks as though Kepler was right! You should now feel confident enough to throw your graph paper away and use the formula instead.

ITQ 26 Trust the dimensional argument! The only way that you can make the dimensions on both sides of the equation balance, is for you to acknowledge that the *constant* in the equation must also have dimensions. Thus, writing the equation in terms of its dimensions:

$$[\text{time}]^2 = [\text{dimensions of the constant}] \times [\text{length}]^3$$

This equation balances if the dimensions of the constant are $[\text{time}]^2/[\text{length}]^3$, since the dimensions of the right-hand side can then be written as:

$$\frac{[\text{time}]^2}{[\text{length}]^3} \times [\text{length}]^3 = [\text{time}]^2$$

the same as on the left-hand side.

The *units* of the constant (as used in ITQ 24) are (Earth-orbital period)^2/(Earth-orbital radii)^3. You should now go back and make sure that, when you answered ITQ 24, you filled in the units correctly in Table 8. (See Table 11 in the ITQ 24 answer.)

If you are still not happy with this concept of 'dimensional analysis', you should refer to the further explanation and examples given in HED. **HED**

TABLE 12

moon	R (km)	T (hours)	R^3 (km)^3	T^2 (hours)^2	T^2/R^3 (hours^2/km^3)
Io	4.22×10^5	42.4	7.52×10^{16}	1.80×10^3	2.39×10^{-14}
Europa	6.71×10^5	85.2	3.02×10^{17}	7.26×10^3	2.40×10^{-14}
Ganymede	10.71×10^5	171.7	1.23×10^{18}	2.95×10^4	2.40×10^{-14}
Callisto	18.84×10^5	400.5	6.69×10^{18}	1.60×10^5	2.39×10^{-14}

ITQ 27 Table 12 shows values of R^3, T^2 and T^2/R^3. As you can see, T^2/R^3 is roughly the same for all four moons of Jupiter. Its value is about $(2.4 \times 10^{-14}) \dfrac{\text{hours}^2}{\text{km}^3}$

You have already found that the constant of proportionality for Kepler's law, as applied to the planets of the solar system, is:

$$1.00 \frac{(\text{Earth years})^2}{(\text{Earth-orbit radii})^3} \quad \text{(see ITQ 24)}$$

If we convert this constant to units of (hours)^2/(km)^3, we can compare it with the constant obtained for Jupiter's moons. To convert the constant, proceed as follows:

$$\text{one Earth year} = (365\tfrac{1}{4} \times 24)\,\text{hours}$$
$$= 8\,766\,\text{hours}$$

$$\text{one Earth-orbital radius} = 1.50 \times 10^8\,\text{km} \quad \text{(see Appendix)}$$

Therefore

$$1.00 \frac{(\text{Earth years})^2}{(\text{Earth-orbital radii})^3} = \frac{1.00 \times (8\,766)^2\ \text{hours}^2}{(1.50 \times 10^8)^3 \quad \text{km}^3}$$

$$= 2.28 \times 10^{-17} \frac{\text{hours}^2}{\text{km}^3}$$

So Kepler's third law *does* apply to the moons of Jupiter, but with a different constant of proportionality from that obtained for the planets in the solar system. (The difference is approximately a factor of 1 000.) You might guess that the change of constant has something to do with the change of orbital-centre from Sun to Jupiter. But that's for Unit 3 to explain.

Acknowledgements

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